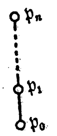
Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 173–182

Fibre Products of Noetherian Rings

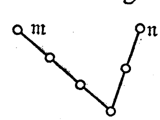
Tetsushi Ogoma

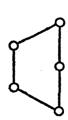
This talk is about fibre product of noetherian rings. The first half is a commentary of the results already published elsewhere, and the second half is new, hence we will give the proof.

Now let me begin with the chain problem of prime ideals. Recall that a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_n$ is saturated if there is no proper prime ideal between $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ for any i $(0 \leq i \leq n-1)$ and we draw a picture of it as the following 

. The length of the chain is n. We say that a noetherian ring A is catenary if for any pair $\mathfrak{p} \subseteq \mathfrak{q}$ of prime ideals in A, the lengths of all saturated chains between $\mathfrak{p}$ and $\mathfrak{q}$ are the same. A noetherian ring A is universally catenary if any A-algebra of finite type is catenary.

The theorem that a geometric ring (an algebra of finite type over a field) is (universally) catenary is a classical result and it was a problem whether every noetherian ring is (universally) catenary. Nagata constructed counter examples to the problem for the first time, and his construction is suggestive and useful to our topic of today. So let us review:

Nagata's example ([9, Example 2]). Let K be a field and take a regular semilocal K-algebra domain $(R, \mathfrak{m}, \mathfrak{n})$ such that $\dim R_{\mathfrak{m}} > \dim R_{\mathfrak{n}}$ and that $K \simeq R/\mathfrak{m} \simeq R/\mathfrak{n}$. Of course, we need an idea to construct such an example and really he had, this is the first point. The second point is to take the subring $A = K + (\mathfrak{m} \cap \mathfrak{n})$ of R. What happens in the process of taking A from R? A picture of typical maximal chain in R looks like 

and that in A looks like , identifying $\mathfrak{m}$ with $\mathfrak{n}$.

Received October 28, 1985.