Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 167-171

# On the Definition of a Euclid Ring

**M. Nagata**

There are several definitions of the notion of a Euclid ring, and we start with historical survey of these definitions. In this article, we mean by a ring a commutative ring with identity, and we propose the following definition:

A ring $R$ is a Euclid ring if there is a pair of an ordered set $W$ with minimum condition and a mapping $\varphi$ of $R$ into $W$ satisfying the condition that for $a, b \in R$, there are $q, r \in R$ such that

$$b=qa+r \text{ with either } r=a \text{ or } \varphi r<\varphi a.$$

This is a modified version of the one which was given by Nagata [3]. The definition given by Samuel [4] is more general than the classical definition and is more restrictive than ours. As was shown by Nagata [2], there is an integral domain which is a Euclid ring in the sense of Samuel, but not in the classical sense. Thus, Samuel's definition is essentially more general than the classical one. But, our new definition does not increase the family of Euclid rings than Samuel's definition, though the choice of algorithm surely enlarges by our generalization.)

We would like to discuss advantage of our new definition, including our proof of the following fact:

The direct sum of a finite number of Euclid rings is again a Euclid ring.

## § 1. Historioal survey

The classical definition of a Euclid ring can be stated as follows (see, for instance, van der Waerden [5]):

An integral domain $R$ is a Euclid ring if there is a mapping $\varphi$ of $R-\{0\}$ into the set $N$ of natural numbers which satisfies two conditions

(1) if $a, b$ are non-zero elements of $R$ then $\varphi(ab)\geq\varphi a$, and

(2) if $a, b \in R$ and $a\neq 0$, then there are $q, r \in R$ with

$$b=qa+r \text{ and either } r=0 \text{ or } \varphi r<\varphi a.$$

Received September 17, 1985.