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# On the Injective Envelope of the Residue Field of a Local Ring

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**0.** In this note a ring will mean a commutative noetherian ring. When we say '$(A, m, k)$ is a local ring', we mean that $A$ is a local ring, $m$ is its maximal ideal and $k$ is its residue field.

**1.** Let $A$ be a ring and $M$ be an $A$-module. The injective envelope $E=E_A(M)$ of $M$ is defined by the following properties:

(1) $E$ contains $M$ as a submodule, and $E$ is an essential extension of $M$.

(2) $E$ is an injective $A$-module.

The first condition is usually easy to check, while the second is sometimes not so easy to verify.

When $(A, m, k)$ is a local ring, $E:=E_A(k)$ has the following remarkable property: if $N\neq 0$ is an $A$-module (not necessarily finitely generated), then $\mathrm{Hom}_A(N, E)\neq 0$. (To see this, take a non-zero element $x$ of $N$ and set $I:=\mathrm{ann}(x)$. Then there is a non-zero $A$-linear mapping $Ax\simeq A/I\to A/m=k\to E$; extend it to an $A$-linear mapping $N\to E$.) Moreover, since $E$ is an essential extension of $k$, it is easy to see that $\mathrm{Ass}_A(E)=\{m\}$, and consequently every element of $E$ is killed by a suitable power of $m$, so that $E$ can be viewed as a module over the completion $\hat{A}$ of $A$. Matlis ([1] Theorem 3.7) showed that $\mathrm{Hom}_A(E, E)=\hat{A}$, in other words every endomorphism of the $A$-module $E$ is realized by multiplication by exactly one element of $\hat{A}$. In particular, $E$ is a faithful $\hat{A}$-module (i.e. $a\in\hat{A}$, $aE=0\Rightarrow a=0$). Using these facts we obtain the following characterization of $E_A(k)$ when $A$ is complete.

**Theorem 1.** *Let $(A, m, k)$ be a complete local ring and $E$ be an $A$-module containing $k$ as submodule. Then $E$ is the injective envelope of $k$ if and only if*

(a) *$E$ is an essential extension of $k$, and*

(b) *$E$ is a faithful $A$-module.*

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