Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 147-160

# Littlewood's Formulas and their Application to Representations of Classical Weyl Groups

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## Introduction

The reciprocity between the representations of the general linear groups and the symmetric groups is well known. For example, in I.G. Macdonald's book [M], this reciprocity is described as a ring isomorphism between the ring $\Lambda$ of symmetric functions in countably many variables (see [M], [K–T]) and the graded ring $R=\oplus_n R(\mathfrak{S}_n)$, where $R(\mathfrak{S}_n)$ is the free $\boldsymbol{Z}$-module generated by the irreducible characters of the symmetric group of degree $n$ and the multiplication in $R$ is defined for $f\in R(\mathfrak{S}_n)$ and $g\in R(\mathfrak{S}_m)$ by $f\cdot g=\mathrm{ind}_{\mathfrak{S}_n\times\mathfrak{S}_m}^{\mathfrak{S}_{n+m}}(f\times g)$. In an analogous manner, we define a graded ring $R_W=\oplus_n R(W(B_n))$ using the characters of the Weyl groups $W(B_n)$ of type $B_n$ and a homomorphism from this ring $R_W$ to $\Lambda$. This homomorphism clarifies the relationship between the representations of $GL(n)$ and the rule of decomposition (into irreducible constituents) of the representations of $\mathfrak{S}_{2n}$ induced by an irreducible representation of $W(B_n)$. In this procedure, Littlewood's formulas play a crucial role. Here, Littlewood's formulas mean the expansion formulas of the following four symmetric rational functions into Schur functions:

(1) $\displaystyle\prod_{1\leqq i<j\leqq n}(1-z_iz_j)^{-1}$,

(2) $\displaystyle\prod_{1\leqq i\leqq j\leqq n}(1-z_iz_j)^{-1}$,

(3) $\displaystyle\prod_{1\leqq i<j\leqq n}(1-z_iz_j)$,

(4) $\displaystyle\prod_{1\leqq i\leqq j\leqq n}(1-z_iz_j)$.

These formulas are also essential in describing the relations between the representations of $GL(n)$ and those of $Sp(2n)$ and $SO(n)$ (see [K-T]).

## § 1. Littlewood's formulas

The four rational functions listed in the introduction are all $\mathfrak{S}_n$-invariant (where $\mathfrak{S}_n$ acts by the permutations of variables $\{z_i\}_{i=1}^n$), There-

Received October 28, 1985.
* Supported by Grant Aid for Scientific Research.