Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 93–109

Distributive Lattices, Affine Semigroup Rings and Algebras with Straightening Laws

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Summary. A lattice L is called integral over a field k if there exists a homogeneous ASL (algebra with straightening laws) domain on L over k. By virtue of fundamental structure theorem of Birkhoff, we can prove that every finite distributive lattice is integral.

Introduction

What properties of a finite poset (partially ordered set) H guarantee the existence of an ASL domain on H over a field k? This is an interesting question lying between commutative algebra and combinatorics.

Investigate many concrete examples of ASL which appeared in classical invariant theory, and we may well hope that every ASL which is an integral domain should have some good properties. If we will try to analyze some ring-theoretical properties of ASL domains, then the above question should arise of necessity.

For example, it is a starting point of recent works [12] and [13], in which the final goal is to classify all the three dimensional homogeneous Gorenstein ASL domains, to determine all the posets on which there exist three dimensional homogeneous Gorenstein ASL domains.

The main purpose of this paper is first to construct an ASL domain $\mathscr{R}_k[D]$ (see § 2) on any finite distributive lattice D over a field k, secondly to calculate the canonical module of $\mathscr{R}_k[D]$ explicitly and give a combinatorial interpretation to the number of minimal generators of this module, and thirdly to determine what kind of distributive lattices are Gorenstein.

This article is divided into four sections. In Section 1, we recall some fundamental definitions and terminologies on commutative algebra and combinatorics. In addition, we shall remark that the tensor product $\mathscr{R}_1 \otimes_k \mathscr{R}_2$ of two ASL's $\mathscr{R}_1$ and $\mathscr{R}_2$ over a field k is again an ASL in a natural way. This result will be used in Section 4. Moreover, for each finite poset H, we will associate a positive integer $t(H)$ and show that $t(H)=1$ if and only if H is pure. This number $t(H)$ will play an essential role in Section 3.

Received October 28, 1985.