Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 21-28

Resolutions and Representations of $GL(n)$

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In his 1890 paper, [7], Hilbert "finished off" invariant theory by proving his Basis Theorem and his Syzygy Theorem. Recall that the syzygy theorem, i.e. the statement that every homogeneous ideal in the polynomial ring $k[X_1, \cdots, X_n]$ has a free resolution of length not greater than $n-1$, was proved to facilitate the calculation of the number, $H(d)$, of independent forms of degree d in the ring of invariants of a linear group action. (It also shows quite trivially that the function $H(d)$ is a polynomial function.) It is therefore very satisfying to see how, almost one hundred years later, we can turn things around and use representation theory to facilitate the calculation of explicit free resolutions of large classes of ideals (and modules). The aim of this paper is to illustrate how this is done in some special cases. In no case will we do a complete computation, but merely do enough to show how to employ the technique. It will become clear that a good computer program involving the Littlewood-Richardson rule as well as the illustrated counting process would be of enormous help.

§ 1. The generic $m \times n$ matrix

The generic $m \times n$ matrix $X=(X_{ij})$ with $1 \leq i \leq m$, $1 \leq j \leq n$, $m \geq n$, is regarded as a map from a free module of rank m to one of rank n. A number of important ideals and modules are associated with this matrix: The ideals I_p generated by the minors of X of order p; the various exterior powers of the cokernel of this linear map.

In order to consider this situation in a more intrinsic way, let us suppose we are working over a field, R, of characteristic zero, and that F and G are vector spaces of dimensions m and n respectively. We may then construct the symmetric algebra $S(F \otimes G)=S$ over R, and let $\bar{F}=S \otimes_R F$, $\bar{G}=S \otimes_R G$. To define a homogeneous map $\phi: \bar{F} \to \bar{G}^*$ of degree 1, it suffices to define a map $\phi_0: F \to S_1 \otimes_R G^*$, where S_1 stands for the component of degree 1 in the graded ring $S=S(F \otimes G)$, i.e. $S_1=F \otimes G$.

Received November 28, 1985.
* Partially supported by NSF Grant No. MCS 83 03348.