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Some Cohen-Macaulay Complexes arising in Group Theory

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In this brief survey we will review some main facts known about the ring-theoretic properties of three classes of simplicial complexes which naturally arise in finite group theory. Ring-theoretic concepts are applied to a finite simplicial complex Δ via its Stanley-Reisner ring $\underset{\sim}{k}[\Delta]$. In matters of definitions and notation we adhere to Stanley's book [20], to where the reader is referred for all unexplained terminology.

§ 1. Subgroup lattices

Let G be a finite group, which we assume not to be of prime order to avoid trivialities. The proper subgroups H of G (i.e., $H \neq G$, 1) are the vertices of a simplicial complex $\Delta(L_G)$ whose faces are the chains $H_0 \subset H_1 \subset \cdots \subset H_k$. Thus, $\Delta = \Delta(L_G)$ is a simplicial complex on which G acts by conjugation, so also the ring $\underset{\sim}{k}[\Delta]$ has an induced G-action.

The following characterization of Cohen-Macaulayness is obtained by combining results of Björner, Iwasawa and Stanley, see [3, § 3].

$\Delta(L_G)$ is Cohen-Macaulay if and only if G is supersolvable.

The smaller class of Gorenstein subgroup lattices and the larger class of Buchsbaum subgroup lattices have also been determined. The next result is due to Hibi [14].

$\Delta(L_G)$ is Gorenstein if and only if G is cyclic of prime-power or square-free order.

The following was shown by Björner and Smith [6].

$\Delta(L_G)$ is Buchsbaum if and only if G is either supersolvable or else a semidirect product of an elementary Abelian group N of order p^k, $k \geq 2$, and a cyclic group of order q acting irreducibly on N, for distinct primes p and q.

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