Advanced Studies in Pure Mathematics 11, 1987
Commutative Algebra and Combinatorics
pp. 9-11

Fibre Rings and Polynomial Rings

Teruo Asanuma

Let R be a noetherian ring. In their joint work [10], Weisfeiler and Dolgachev study the structure of R-algebra A satisfying the following condition: The fibre ring $A \otimes k(\mathfrak{p})$ is always isomorphic to a polynomial ring in n variables over $k(\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ of R where $k(\mathfrak{p})$ denotes the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. They investigate such an A and ask what conditions on A guarantee that A must be a polynomial ring. In particular they conjecture that A must be a polynomial ring in the case where A is finitely generated flat over a normal local domain R. This conjecture has been settled affirmatively by Waterhouse [9] provided that A is a ring of functions on a group scheme over a discrete valuation ring R. Without assumption on group structure, the conjecture is true for $n=1$ [1] (See also [5] and [7]). The case $n=2$ has been proved for R a discrete valuation ring by Sathaye after the Kambayashi's contribution [6] with the additional hypothesis that R contains the rational number field $\boldsymbol{Q}$. On the other hand there is a counter example to the conjecture due to Swan and Yanik [11] as is pointed out by Eakin in [4] in the case where R is not a valuation ring.

Now we will discuss the stable structure of A, where the stable structure means the structure of a polynomial ring $A^{[m]} := A[x_1, \cdots, x_m]$ in m variables as an R-algebra for some large integer m. From this point of view we have first the following theorem.

Theorem 1. *Let R be a discrete valuation ring with quotient field K and residue field k. Let A be an integral R-domain such that $A \otimes K \cong K^{[n]}$ and $A \otimes k \cong k^{[n]}$. Then $A^{[n]} \cong R^{[2n]}$.*

Two R-algebras B and C are called stably isomorphic if $B^{[m]} \cong C^{[m]}$ for some positive integer m. So the theorem shows that A is stably isomorphic to $R^{[n]}$ in the case where A is an integral domain and R is a discrete valuation ring. By virtue of the theorem we can delete the hypothesis "finitely generated" from the Sathaye-Kambayashi's result. However we can not delete the additional hypothesis $R \supset \boldsymbol{Q}$ as follows:

Received October 30, 1985.