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Representations of $GL(n)$ and Schur Algebras

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The fundamental representations of the algebraic group $GL(n)$ are the exterior powers $\Lambda^t(V)$ of the standard module V of dimension n and their formal characters are the elementary symmetric polynomials $e_t(x_1, \cdots, x_n)$ in n variables. The formal character of the Schur module $L_\lambda(V)$ for a partition $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_m)$ is a symmetric polynomial known classically as the Schur function $\{\tilde{\lambda}\}$ for the transpose partition $\tilde{\lambda}$. The symmetric polynomial $\{\tilde{\lambda}\}$ can be expressed in terms of the elementary ones as the determinant of the m by m matrix whose (i, j)-th entry is the elementary symmetric polynomial of degree $\lambda_i - i + j$. This expansion corresponds to the Giambelli expansion for a Schubert cycle on a Grassmannian in terms of the special Schubert cycles and it is actually just the transpose of the usual Jacobi-Trudi expansion for a Schur function. More will be said about the Jacobi-Trudi identity later.

There is a useful description of the above expansion for $\{\tilde{\lambda}\}$ in terms of the twisted action of the symmetric group on sequences. For any sequence $\gamma=(\gamma_1, \cdots, \gamma_m)$ of integers it is customary to let e_γ denote the monomial $e_{\gamma_1} \cdots e_{\gamma_m}$ of elementary symmetric polynomials. The expansion then can be written in the form

$$\{\tilde{\lambda}\} = \sum (-1)^w e_{w\cdot\gamma} \tag{1}$$

where the summation is over all permutations w of $\{1, \cdots, m\}$ and $w \cdot \lambda$ denotes the twisted action $w(\lambda+\zeta)-\zeta$ with $\zeta=(m-1, m-2, \cdots, 2, 1, 0)$. If for any sequence $\lambda=(\lambda_1, \cdots, \lambda_m)$ we let $\Lambda_\gamma(V)$ denote the tensor product representation $\Lambda^{\gamma_1}(V) \otimes \cdots \otimes \Lambda^{\gamma_m}(V)$ then the identity in (1) can be written as

$$[L_\lambda(V)] = \sum (-1)^w [\Lambda_{w\cdot\lambda}(V)] \tag{2}$$

in the formal character ring, or Grothendieck ring, of polynomial representations of $GL(n)$.

It was observed by A. Lascoux that the above identity should be realized as a resolution

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