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Mixed Hodge Structures on Cohomologies with Coefficients in a Polarized Variation of Hodge Structure

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Introduction

Let U be a smooth quasi-projective variety over $\boldsymbol{C}$ and V a polarized variation of Hodge structure of weight m on U (cf. (0.3), [Gr1], [S]). A basic problem in the Hodge theory is to show the existence of a canonical mixed Hodge structure on the cohomology $H^i(U, V)$.

As is well known, the constant coefficient case $V=\boldsymbol{Q}_U$ is just Deligne's mixed Hodge theory [D2]. The case dim $U=1$ was treated by Zucker [Z1].

In this paper, we try to generalize Zucker's result to higher dimension in a special case. We assume U to be the complement of a smooth hypersurface Y in a projective smooth variety X (cf. (0.1)):

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y = X - U.$$

Under this assumption, we obtain a slight generalization of Zucker's theorems (cf. [Z1, (7.12), (13.11), (14.3)]).

Theorem 1 (cf. (2.5)). *There exists a natural pure Hodge structure of weight $m+i$ on $H^i(X, j_* V)$.*

Theorem 2 (cf. (3.1.7), (3.2.6)). *There exists a natural mixed Hodge structure of weight $\geqq m+i$ (resp. $\leqq m+i$) on $H^i(U, V)$ (resp. $H^i_c(U, V)$).*

Such a generalization of [Z1] was obtained independently by Zucker himself, as announced in [Z2, p. 182].

We shall indicate the proof of these theorems, which is reduced to the one-dimensional case [Z1] by our assumption on Y (cf. (0.5)).

For Theorem 1, we use the Hodge theory for V-valued L^2-forms in Section 2, in parallel with [Z1]. Once an L^2-complex is constructed, its properties are proved by means of local Künneth type formulas. Note that the norm estimate due to Schmid [S, (6.6)] is essential here.

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