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The Lower Semi-Continuity of the Plurigenera of Complex Varieties

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Introduction

This paper is an extension of [34] in which it was shown that the **Conjecture** L (see below) follows from the minimal model conjectures in the case of algebraic varieties. In this paper, we treat complex varieties.

Conjecture L. *Let $\pi: X \to D$ be a proper surjective morphism from a complex manifold X onto a unit disk D. Assume that $\pi^{-1}(0)=\bigcup_{i\in I}\Gamma_i$, where all the Γ_i are compact complex varieties in class $\mathscr{C}$ in the sense of Fujiki* [5]. *Then*

$$\sum_{i\in I} P_m(\Gamma_i) \leqq \operatorname{rank} \pi_* \mathcal{O}_X(mK_X) \qquad \textit{for all } m \geqq 1,$$

where P_m denotes the m-genus.

Clearly, this induces the invariance of plurigenera under smooth deformations. The invariance of the plurigenera of compact complex surfaces was proved by Iitaka [16]. But we have many counterexamples without assuming that the Γ_i belong to the class $\mathscr{C}$ in the higher dimensional case or even in the case of degeneration of surfaces (see Nakamura [31], Nishiguchi [36]).

The theory of minimal models developed by Mori, Reid, Kawamata, Tsunoda, Shokurov, Benveniste, Kollár and others is not yet completed even in the case of algebraic varieties. In this paper we shall prove Conjecture L in the case of semi-stable relative minimal models. A relative good minimal model $\pi: X \to D$ is defined to be a proper surjective morphism from a variety X with only canonical singularities such that K_X is π-semi-ample. Conjecture L can be proved with the help of some kind of the theory of minimal models. In fact if π is a projective degeneration of surfaces with non-negative Kodaira dimensions, then it is proved (see (7.5)) by a result of Tsunoda [48]. The main technique of our paper is the same one as in Kawamata [21]. But since his arguments require some properties of projective varieties in some steps, we must modify the proofs.

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