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Cremona Transformations and Degrees of Period Maps for $K3$ Surfaces with Ordinary Double Points

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Let X be a $K3$ surface with n ordinary double points on which an ample line bundle $\mathscr{L}$ has been fixed. If $\mu: S \to X$ is the minimal desingularization, the orthogonal complement in $H^2(S, \boldsymbol{Z})$ of $\mu^*\mathscr{L}$ and the rational curves $\mu^{-1}(P)$ (for $P \in \operatorname{Sing} X$) carries a Hodge structure with $h^{2,0} = h^{0,2} = 1$ and $h^{1,1} = 19 - n$. The *period map* for these surfaces is the natural map from the moduli space to the classifying space for such Hodge structures; this generalizes the classical period maps for polarized $K3$ surfaces [24].

In contrast to the classical case, these period maps, while always étale, tend to have degree greater than one. In this paper, we study this phenomenon for two particular kinds of $K3$ surfaces with ordinary double points: the "$K3$ surfaces of Cremona type" in which the degree of $\mathscr{L}$ is 2 and the branch locus of the induced map to $\boldsymbol{P}^2$ is irreducible, and the "$K3$ surfaces of Todorov type" whose study was begun in [20]; these latter surfaces arise as quotients of certain surfaces of general type constructed by Todorov [26]. (We omit one of the families of $K3$ surfaces of Todorov type here, as to consider it would take us too far afield.)

Our main results are a computation of the degrees of the period maps (Corollaries (5.2) and (5.6)), a demonstration that for our families, two $K3$ surfaces with the same periods are birationally (but not always biregularly) isomorphic (Theorem (6.1)), and finally a consideration of the geometric consequences of these birational isomorphisms when the degree of $\mathscr{L}$ is small (Theorems (7.1), (7.3), and (8.5)). The geometric consequences we find involve the behavior of sets of points in $\boldsymbol{P}^2$ or $\boldsymbol{P}^3$ under the Cremona group of birational automorphisms of $\boldsymbol{P}^2$ or $\boldsymbol{P}^3$ (hence the name "Cremona type"), and we obtain modern proofs of some classical results of Coble [3], [4], [5] on this topic. We should mention that Coble's work has recently been studied from a different point of view by Cossec and Dolgachev [8], [6], whose results we use in our interpretation.

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