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Projective Degenerations of Surfaces according to S. Tsunoda

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§ 1. Introduction

This is a partial and incomplete account of a work of S. Tsunoda on projective degenerations of algebraic surfaces. We shall begin with some necessary definitions. In the following, we take the complex field $\boldsymbol{C}$ as the ground field.

Degenerations of algebraic surfaces as considered in Persson [9] mean the surfaces $\mathscr{X}_0$ appearing in the following setup:

Let $\pi: \mathscr{X} \to \Delta$ be a proper, flat morphism of a smooth threefold X defined over $\boldsymbol{C}$ onto a small open disc Δ such that π is smooth over $\Delta^* = \Delta - \{0\}$. Then $\mathscr{X}_0 = \pi^{-1}(0)$. Namely, $\mathscr{X}_0$ is thought of as a degeneration of a general smooth fiber $\mathscr{X}_t$ $(t \neq 0)$.

In considering degenerations of algebraic surfaces, it is standard, in view of Mumford's theorem on semistable reduction, to treat the case where the fiber $\mathscr{X}_0$ is a reduced divisor with simple normal crossings, i.e., $\pi: \mathscr{X} \to \Delta$ is a semistable degeneration.

The first important contribution was made by Kulikov [5], where he considered the degenerations of $K3$ surfaces and he employed essentially an operation of non-algebraic nature, called "a generic contraction". The same subject was taken up later by Persson-Pinkham [10]. Their result says that:

If $\pi: \mathscr{X} \to \Delta$ is a semistable degeneration of algebraic surfaces such that a general fiber $\mathscr{X}_t$, $t \neq 0$ has trivial canonical divisor and that all components of $\mathscr{X}_0$ are algebraic, then there exists a semistable modification $\pi': \mathscr{X}' \to \Delta$ of π such that the canonical divisor of the total space $\mathscr{X}'$ is trivial.

D. Morrison [8] and Tsuchihashi [11] considered the degenerations of Enriques' and hyperelliptic surfaces.

On the other hand, concerning the construction of a minimal model in dimension three, one can consider the following problem (cf. [4]).

Given a nonsingular projective threefold X of non-negative Kodaira

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