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Subadditivity of the Kodaira Dimension: Fibers of General Type

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I. Introduction

The aim of classification theory of algebraic varieties is to exhibit the order in the behavior of algebraic varieties. From this point of view, curves were well understood already in the nineteenth century—surfaces turn out to be much more complicated, and it is only recently that their theory achieved a satisfactory level of completeness. The outlines of an order among higher dimensional varieties has only started to emerge.

One possible approach is to define good numerical invariants of varieties and then use these invariants to relate varieties. One possible candidate is $h^0(\omega_X)$ where ω_X is the dualizing sheaf of a smooth projective variety X. For curves this invariant contains all the "discrete" information about the variety, but already surfaces with $h^0(\omega_X)=0$ have very little in common. It seems better to consider the asymptotic behavior of the numbers $P_m(X)=h^0(\omega_X^m)$. There is a largest number k such that $0<\limsup P_m(X)m^{-k}<\infty$; this k is called the Kodaira dimension of X (denoted by $\kappa(X)$). We put $\kappa(X)=-\infty$ if $P_m(X)=0$ for every m. $\kappa(X)$ is an integer and can take the values $-\infty, 0, 1, \cdots, n$ for n-dimensional varieties. Varieties satisfying $\kappa(X)=\dim X$ are said to be of general type.

One of the basic problems about the behavior of the Kodaira dimension was formulated by Iitaka [I].

Conjecture. *Let $f: X\to Y$ be a surjective map between smooth projective varieties and let F be the generic fiber of f. Then*

$$\kappa(X)\geqslant\kappa(F)+\kappa(Y);$$

i.e., the Kodaira dimension is subadditive for algebraic fiber spaces.

So far various special cases of this conjecture have been proved. The most important cases when an affirmative answer was obtained are the following:

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