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Supersingular Abelian Varieties of Dimension Two or Three and Class Numbers

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§ 0. Introduction

Let B be a definite quaternion algebra over the field Q of rational numbers with discriminant p. We assume that p is a prime number. Let U_g be the positive definite quaternion hermitian space of dimension g over B. We denote by H_g the class number of the principal genus of U_g (for the definition of the principal genus, see Hashimoto and Ibukiyama [6, Section 1]). Let k be an algebraically closed field of characteristic p. In Ibukiyama, Katsura and Oort [7, Theorem 2.10], we showed that the class number H_g $(g \geq 2)$ is equal to the number of isomorphism classes of principally polarized abelian varieties (X, Θ) of dimension g defined over k such that X is isomorphic to a product of supersingular elliptic curves (see also Shioda [23, Theorem 3.5] and Serre [22]).

When $g=1$, H_1 is nothing but the class number of the maximal orders of B. The explicit formula for H_1 was given by Eichler [3, Satz 2]. Then, Deuring proved that the class number H_1 is equal to the number of isomorphism classes of supersingular elliptic curves defined over k (Deuring [2], p. 266). Finally, Igusa calculated the number of isomorphism classes of supersingular elliptic curves defined over k by an algebraic method, and using Deuring's result, he gave a new proof of the explicit formula for H_1 (cf. Igusa [8]). In the first part of this paper, we calculate, by an algebraic method similar to the one in Igusa [8], the number of isomorphism classes of principally polarized abelian surfaces defined over k such that X is isomorphic to a product of supersingular elliptic curves. Hence, using the above result, we give a new proof of the explicit formula for H_2 which was given in Hashimoto and Ibukiyama [6].

Recall that an abelian variety is called supersingular if it is isogenous to a product of supersingular elliptic curves. Let $\mathscr{A}_{g,1}$ be the coarse moduli scheme of principally polarized abelian varieties of dimension g defined over k, and let V be the algebraic set in $\mathscr{A}_{g,1}$ whose points correspond to supersingular abelian varieties. We call V the supersingular locus

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