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On p-Adic Vanishing Cycles
(Application of ideas of Fontaine-Messing)

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§ 0. Introduction

Let K be a complete discrete valuation field with perfect residue field k such that $\mathrm{char}(K)=0$ and $\mathrm{char}(k)=p>0$. Let X be a smooth scheme over the valuation ring O_K of K, and fix notations as

$$X_K = X\otimes_{O_K} K, \qquad Y = X\otimes_{O_K} k,$$
$$X_{\bar{K}} = X_K\otimes_K \bar{K}, \qquad \bar{Y} = Y\otimes_k \bar{k}, \qquad \bar{X} = X\otimes_{O_K} O_{\bar{K}},$$
$$Y \xrightarrow{i} X \xleftarrow{j} X_K, \qquad \bar{Y} \xrightarrow{\bar{i}} \bar{X} \xleftarrow{\bar{j}} X_{\bar{K}}$$

where $\bar{K}$ (resp. $\bar{k}$) denotes the algebraic closure of K (resp. k) and $O_{\bar{K}}$ denotes the integral closure of O_K in $\bar{K}$.

The sheaf of p-adic vanishing cycles $\bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z})$ was studied in [3] [4] and used for the study of the etale cohomology group $H^*(X_{\bar{K}}, \boldsymbol{Z}/p^n\boldsymbol{Z})$. The purpose of Chapter I of this paper is to prove the following new result concerning $\bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z})$. In Section 1, we define certain complexes $\mathcal{S}_n(r)_{\bar{X}}$ ($n\geq 1, 0\leq r<p$) on $\bar{Y}_{\mathrm{et}}$ which come from the crystalline cohomology theory, following the ideas of J.-M. Fontaine and W. Messing in their "syntomic cohomology theory" ([10]).

Theorem (Ch. I (4.3)). *Let $0\leq r<p-1$. Then for any $n\geq 1$, there is a canonical isomorphism*

$$\mathcal{H}^q(\mathcal{S}_n(r)_{\bar{X}}) \cong \begin{cases} \bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z}(r)) & \text{if } q\leq r \\ 0 & \text{if } q>r. \end{cases}$$

Here $\boldsymbol{Z}/p^n\boldsymbol{Z}(r)$ denotes the Tate twist of $\boldsymbol{Z}/p^n\boldsymbol{Z}$. By the definition of $\mathcal{S}_n(r)_{\bar{X}}$ in Ch. I § 1, the result means that $\bar{i}^*R^q\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z})$ can be described in terms of differential forms.

In Chapter II, we shall apply the above result to obtain the following theoerm on the etale cohomology groups $H^q(\bar{Y}, \bar{i}^*R^r\bar{j}_*(\boldsymbol{Z}/p^n\boldsymbol{Z}))$ in the case

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