Advanced Studies in Pure Mathematics 10, 1987
Algebraic Geometry, Sendai, 1985
pp. 85–90

Euler Characteristics and Swan Conductors

Spencer J. Bloch

The purpose of this note is to describe some recent work on Euler characteristics in degenerating families of curves, with particular emphasis on mixed characteristic degenerations. Let $S=\mathrm{Spec}(\Lambda)$ be the spectrum of a complete discrete valuation ring with algebraically closed residue field. Write s (resp. η, resp. $\bar{\eta}$) for the closed (resp. generic, resp. geometric generic) point of S. Let $f: X \to S$ be flat and proper with fibre dimension 1. We assume X is regular and the generic fibre $X_\eta \to \eta$ is smooth. For z either s or $\bar{\eta}$, let

$$\chi(X_z)=\sum(-1)^i \dim H^i_{et}(X_z, \boldsymbol{Q}_l),$$

the étale Euler characteristic of the corresponding fibre. We ask for a formula calculating $\chi(X_s)-\chi(X_{\bar{\eta}})$.

In characteristic zero ($\Lambda=\boldsymbol{C}[[t]]$) the result is understood (cf. [2] ex. 14.1.5); dt gives a section of the sheaf $\Omega^1_{X/\boldsymbol{C}}$ of Kähler differentials, to which one can associate a localized chern class $\boldsymbol{Z}(s_f) \in \mathrm{CH}_0(X_s)$. (I will follow the notation in Fulton's book op. cit. except that I prefer to denote the Chow group of dim. n cycles by CH_n.) One gets in this case for X a degenerating family of varieties of any dimension, $\deg \boldsymbol{Z}(s_f)=(-1)^{\dim X}(\chi(X_s)-\chi(X_{\bar{\eta}}))$. Note that one can (and we will) think of $(-1)^{\dim X}\boldsymbol{Z}(s_f)$ as a local contribution to the cycle-theoretic self-intersection of the diagonal.

$$(1) \qquad (\Delta_X \cdot \Delta_X)_s=(-1)^{\dim X}\boldsymbol{Z}(s_f)=\chi(X_s)-\chi(X_{\bar{\eta}}).$$

We will be most interested in the mixed characteristic and pure characteristic p analogues of this result. There are two problems with (1) in these cases. First, in mixed characteristic, our construction of $\boldsymbol{Z}(s_f)$ doesn't make sense. (What is $\Omega^1_{X/\boldsymbol{C}}$?) Second, even in the pure char. p case, the formula is wrong! It is clear, however, from the global *Grothendieck-Ogg-Shafarevich formula* [6] that the appropriate correction factor is the *Swan conductor* (cf. [8] as well as the discussion below)

Received February 10, 1986.
Partially supported by the National Science Foundation and the Japan Society for the Promotion of Science.