Advanced Studies in Pure Mathematics 9, 1986
Homotopy Theory and Related Topics
pp. 335–344.

On the Image $\rho(BP^*(X)\to H^*(X; Z_p))$

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In this paper we study ways to calculate the Brown-Peterson cohomology $BP^*(X)$ localized at a prime p when the Steenrod algebra action on the ordinary mod p cohomology $H^*(X; Z_p)=HZ_p^*(X)$ is known. One of the most difficult problems is to know which elements in $H^*(X)_{(p)}$ are permanent cycles in the Atiyah-Hirzebruch spectral sequence $H^*(X; BP^*)\Rightarrow BP^*(X)$. This is equivalent to know the image ρ: $BP^*(X)\to H^*(X)_{(p)}$ where ρ is the Thom map.

Cohomology operations on $HZ_p^*(X)$ give some informations about the image. For example if $Q_n x\neq 0$ in $HZ_p^*(X)$, then x is not in Image $\rho(BP^*(X)\to HZ_p^*(X))$, where Q_n is the Milnor primitive operation. We study the above facts in more general situation.

Let: ρ: $h\to k$ be a map of spectra. In Section 1, we note the importance of Image $\rho(h^*(k)\to k^*(k))=\rho(h, k)$, indeed, if an operation θ is in $\rho(h, k)$, then for each $x\in k^*(X)$, $\theta x\in$ Image $\rho(h^*(X)\to k^*(X))$. The image $\rho(P(n), P(m))$ and $\rho(k(n), HZ_p)$ are studied in Section 2. Since $\rho(PB, HZ_p)=0$, we consider $K(Z, n)$ or $K(Z_n, n)$ as k instead of HZ_p in Section 3. Here we introduce the Tamanoi's results. In Section 4, $\rho(BP, K(Z, 3))$ and $BP^*(K(Z, 3))$ are studied. Applications for finite H-spaces are given in Section 5. For example, in the case $p=2$, let X be a simply connected finite associative H-space and let Q^* be the indecomposable elements in $HZ_2^*(X)$. Then

$$(Q^{2n+1})^2\subset \text{Image } \rho(BP^*(X)\longrightarrow HZ_2^*(X)/(HZ_2^+(X)^3)).$$

The author thanks to the referee who pointed out this proof of Proposition 2.4 and corrected many errors in the first version.

§ 1. Maps of cohomology theories

Let ρ: $h\to k$ be a map of spectra and let $k=\{k_n\}$ be the Ω-spectrum, i.e., $k^n(X)\simeq[X, k_n]$. For simplicity of notations, let us write Image $\rho(h^*(k)\to k^*(k))$ (resp. Image $\rho(h(k_n)\to k(k_n))$) by $\rho(h, k)$ (resp. $\rho(h, k_n)$).

Received January 18, 1985.