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On the Spectra $L(n)$ and a Theorem of Kuhn

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Dedicated to Professor Nobuo Shimada on his 60th birthday

§0. Introduction

Let Z_2^n be the elementary abelian 2-group. In [7] Mitchell and Priddy have shown that stably BZ_2^n contains some copies of spectra $M(n) = e_n BZ_2^n$ as a direct summand, where $e_n \in \hat{Z}_2 GL_n(F_2)$ is the Steinberg idempotent. It is also shown that there is an equivalence of spectra $M(n) \simeq L(n) \vee L(n-1)$, where $L(n) = \Sigma^{-n} Sp^{2^n} S^0 / Sp^{2^{n-1}} S^0$. In [5], Kuhn has shown that there is a split exact sequence

$$\longrightarrow L(n) \longrightarrow L(n-1) \longrightarrow \cdots \longrightarrow L(0) = S^0$$

extending the Kahn-Priddy theorem [4] and solved the Whitehead conjecture.

In [9], the author determined the structure of the stable homotopy group $\{BZ_2^n, BZ_2^m\}$ and the composition formula. Let $M_{n,m}(F_2)$ be the set of (n, m)-matrices. Then there are inclusions of rings

$$\hat{Z}_2 GL_n(F_2) \longrightarrow \hat{Z}_2 M_{n,n}(F_2) \longrightarrow \{BZ_2^n, BZ_2^n\} \longrightarrow [QBZ_2^n, QBZ_2^n]$$

where $QBZ_2^n = \Omega^\infty \Sigma^\infty BZ_2^n$ is the infinite loop space.

In this paper, studying the structure of those rings we shall show the following. The Steinberg idempotent $e_n \in \hat{Z}_2 GL_n(F_2)$ is decomposed as $e_n = a_n + b_n$ in the bigger rings and a_n, b_n are primitive in $\{BZ_2^n, BZ_2^n\}$. We determine the structure of $\{M(n), M(m)\}$ and $\{L(n), L(m)\}$. Finally we give a simple proof of the theorem of Kuhn.

§ 1. Steinberg idempotents and matrix algebra

Let R be the ring of 2-adic integers $\hat{Z}_2$ or the prime field F_2. Let $M_{n,m}(F_2)$ be the set of all (n, m)-matrices over F_2. We denote by $R\tilde{M}_{n,m}(F_2)$ the free R-module generated by elements of $M_{n,m}(F_2)$ with the relation 0-matrix$=0$. There is an obvious pairing

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