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A Generalization of the Adams Invariant and Applications to Homotopy of the Exceptional Lie Group G_2

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The original e-invariant of Adams [2] is a homotopy invariant of a map between spheres, which is defined in terms of a coefficient in Chern character equations for a cofibre of a given map, and takes values in rationals modulo one. Generalizations in several directions have been obtained including the case of a map between spaces with no torsion in homology and with suitable conditions. In this paper, we present and discuss a variant of the Adams invariant for a map between spaces whose comlex K-groups are isomorphic to those of spheres. Such a space must have torsion in homology unless it is a sphere, and we give, in (1.3), (1.3)′, (1.3)″ below, a class of three-cell complexes which are simplest examples of such spaces.

The theorem of Hodgkin [9] support that there is a class of Lie groups which have the property that their K-groups are free but they have indeed torsion in homology. The first example of our three-cell complexes is, in fact, closely related to the compact, simply connected exceptional Lie group G_2, an example of such Lie groups. Namely, if we put

$$X^n = S^n \cup_\eta e^{n+2} \cup_2 e^{n+3}, \qquad n \geqq 3,$$
$$Y^n = S^{n-3} \cup_2 e^{n-2} \cup_\eta e^n, \qquad n \geqq 6^\dagger,$$

then they have the K-groups isomorphic to those of S^n, and we may define the stable Adams invariant

$$e: \{Y^{n+7}, X^n\} \longrightarrow Q/\tfrac{1}{2}Z$$

($\{Y, X\}$ denotes the group of stable maps from Y to X), which takes values in rationals modulo integral multiples of 1/2, the rational depending only on the source space Y. The relation between the spaces X^n, Y^n and the Lie group G_2 is that, up to homotopy equivalence, there is a CW decom-

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† It is known that Y^5 also exists by the computation of the unstable group $\pi_4(S^2 \cup e^3)$; however, we shall need to use Y^n only for $n \geqq 8$.