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Characteristic Classes of T^2-bundles

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§ 1. Introduction

In our previous paper [5], we have proposed the problem to determine characteristic classes of differentiable fibre bundles whose fibres are diffeomorphic to a given closed manifold M, in other words the problem to compute the cohomology group $H^*(B\,\mathrm{Diff}\,M)$. The case when M is a closed orientable surface of genus greater than or equal to two has been treated in [4] [5]. In this paper we consider the case when M is the 2-dimensional torus T^2. Let $\mathrm{Diff}_+ T^2$ be the group of all orientation preserving diffeomorphisms of T^2 equipped with the C^∞ topology. Then our main result is

Theorem 1.1.

$$\dim \tilde{H}^n(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q}) = \begin{cases} 0 & n \not\equiv 1 \pmod 4 \\ 2m-1 & n = 24m+1 \\ 2m+1 & n = 24m+5,\ 24m+9,\ 24m+13 \\ & \qquad\qquad \textit{or}\ 24m+17 \\ 2m+3 & n = 24m+21. \end{cases}$$

The first non-trivial group is $H^5(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q}) \cong \boldsymbol{Q}$ and $\dim H^{4k+1}(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q})$ is approximately $\frac{1}{3}k$. Obviously the ring structure on $H^*(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Q})$ defined by the cup product is trivial. We can also obtain informations on the torsions and by making use of them we obtain

Theorem 1.2. Mod 2 *and* 3 *torsions, we have*

$$\tilde{H}_n(B\,\mathrm{Diff}_+ T^2; \boldsymbol{Z}) = \begin{cases} \textit{torsion} & n \equiv 0 \pmod 4 \\ \textit{free abelian group of rank} & \\ \textit{indicated in Theorem } 1.1 & n \equiv 1 \pmod 4 \\ 0 & n \equiv 2,\ 3 \pmod 4. \end{cases}$$

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