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Cohomology mod p of the 4-connected Cover of the Classifying Space of Simple Lie Groups

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§0. Introduction

Let G be a compact, connected, simply connected, simple Lie group and BG its classifying space. A prime p is called good (for G) (resp. exceptional (for G)) if $H_*(G; Z)$ is p-torsion free (resp. not p-torsion free). As is well known BG is 3-connected and $\pi_4(BG)=H_4(GB; Z)=H^4(BG; Z)=Z$ (cf. [3]). Represent a generator of $H^4(BG; Z)$ by a map $Q'': BG\to K(Z, 4)$ and denote its homotopy fibre by $B\tilde{G}$. The purpose of this paper is to determine $H^*(B\tilde{G}; F_p)$ for any odd prime p.

Consider the following pull back diagram:

$$\begin{array}{ccccc} K(Z,3) & \xrightarrow{\pi'} & B\tilde{T} & \xrightarrow{Q'} & BT \\ \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \bar{i}} \\ K(Z,3) & \xrightarrow[\pi]{} & B\tilde{G} & \xrightarrow[Q'']{} & BG \end{array}$$

where T is a maximal torus, i and $\bar{i}$ are the maps induced by the inclusion. Note that $\bar{i}^*: H^4(BG; Z)\to H^4(BT; Z)$ is a monomorphism and $\mathrm{Im}\,\bar{i}^*=H^4(BT; Z)^{W(G)}$ where $W(G)$ is the Weyl group of G. Therefore $Q'=\bar{i}^*Q''$ is a generator of $H^4(BT: Z)^{W(G)}$. Denote the mod p reduction of Q' by Q. Since $H^*(BT; F_p)\cong S(H_2(BT, F_p)^*)$, where S denotes the symmetric algebra, we may consider that Q is a quadratic form. Let $h=h(G,p)$ be the codimension of a Q-isotropic subspace of maximum dimension.

As is well known that

$$H^*(K(Z, 3); F_p)\cong S(\beta P_k u_3;\ k\geq 1)\otimes E(P_k u_3;\ k\geq 0)$$

where E denotes the exterior algebra, $P_k=\mathscr{P}^{p^{k-1}}\cdots\mathscr{P}^1$ and u_3 is a generator of $H^3(K(Z, 3); F_p)$ $(=Z/p)$. Denote the subalgebra generated by $\{\beta P_k u_3;\ k\geq 1\}\cup\{P_k u_3;\ k\geq j\}$ by R_j. Then the main results of this paper are the following:

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