Advanced Studies in Pure Mathematics 9, 1986
Homotopy Theory and Related Topics
pp. 75–108

Cohomology of Classifying Spaces

Hiroshi Toda

Introduction

The concept of the classifying space and characteristic classes are great tools in both geometry and topology.

Originally, the classifying space BG appeared as Grassmannian manifolds in discussing the equivalence of the fibre bundles of a fixed structure group G operating effectively on the fibre. And, the equivalence classes of such bundles on a CW-complex X are in one-to-one correspondence naturally with the homotopy classes of maps $f: X \to BG$ [39].

The classifying space BG of a topological group G is characterized as the base space of a universal G-bundle $G \to EG \to BG$ of ∞-connected total space EG. So, up to homotopy type, we may consider that the loop space ΩBG of BG is G, and BG is the de-looping of G.

For every associative H-space G, a classifying space BG is also constructed geometrically, and the construction is applied to give Eilenberg-Moore spectral sequences.

Lately, classifying space appeared in the theory of generalized cohomology. For each Brown functor F, i.e. a functor F satisfying wedge axiom and Mayer-Vietoris axiom, on the category of pointed finite CW-complexes. Under suitable condition, there exists a classifying space Y of F such that the functor F is naturally equivalent to the functor $[-, Y]_0$ of pointed homotopy classes. Then the original classifying space BG is that for the functor taking principal G-bundles over given base space [42].

The characteristic classes of fibre bundles are considered as a natural functor of fibre bundles to a cohomology class of the base spaces. For classical groups there are specially named characteristic classes, the Chern classes $c_n \in H^{2n}$ for unitary groups and complex general linear groups, the Stiefel-Whitney classes $w_n \in H^n(\ ; Z/2)$ for orthogonal groups and real general linear groups, the Pontrjagin classes $p_n \in H^{4n}$ for special orthogonal groups and real special linear groups, and others. By general theory of universal bundles, each characteristic class corresponds to an element of the cohomology $H^*(BG; -)$. The structure of the cohomology ring

Received October 17, 1985.