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The Kervaire Invariant of Some Fiber Homotopy Equivalences

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§ 0. Introduction

Suppose we are given a proper fiber preserving map $\xi: E_+ \to E_-$ of degree d between vector bundles $E_\pm$ over a closed smooth manifold X. Then transversality theorem says that one can convert ξ via a proper homotopy into a map h transverse to the zero section X in E_-. This produces a triple $\kappa=(W, f, \beta)$ called a normal map of degree d; $W=h^{-1}(X)$, $f=h|W: W \to X$ is a degree d map, and $\beta: TW \to f^*(TX+E_+-E_-)$ is a stable vector bundle isomorphism. We perform surgery on κ via normal cobordism to make f a homology (or homotopy) equivalence with an appropriate coefficient. Unfortunately this is not always possible. We encounter an obstruction called surgery obstruction. Needless to say, to know whether or not the surgery obstruction vanishes is a substantial problem which is studied by many people. Among surgery obstructions the Kervaire invariant (or sometimes called the Kervaire obstruction) has a rich history and is an object with outstanding interest. It is defined with a value in $Z/2=\{0, 1\}$ provided that the dimension of X is even and d is odd. Since it depends only on the given map ξ, we will denote it by $c(\xi) \in Z/2$.

In this paper we are concerned with the Kervaire invariant obtained in the following way. Let G be a (compact) Lie group. Suppose we are given a principal G bundle $\pi: P \to X$ and a proper G map $\omega: V \to U$ of an odd degree d between G representations. Then we combine them to get a proper fiber preserving map of degree d, written ω_π, between associated vector bundles $P \times_G V$ and $P \times_G U$. Obviously $c(\omega_\pi)$ must be described in terms of the given data ω and π. The purpose of this article is to give an explicit description in some cases.

As a consequence there exists a fiber homotopy equivalence with the Kervaire invariant one over some X. For instance this is the case for $X=P(C^n)$ (n: even) the complex projective space of complex dimension

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