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A Necessary and Sufficient Condition for a Local Commutative Algebra to be a Moduli Algebra: Weighted Homogeneous Case

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Let $\mathcal{O}_{n+1}=\boldsymbol{C}\{z_0, z_1, \cdots, z_n\}$ denote the ring of germs at the origin of holomorphic functions $(\boldsymbol{C}^{n+1}, 0)\to\boldsymbol{C}$. If $(V, 0)$ is a germ at the origin of a hypersurface in $\boldsymbol{C}^{n+1}$, let $I(V)$ be the ideal of functions in $\mathcal{O}_{n+1}$ vanishing on V, and let f be a generator of $I(V)$. It is well known that $V-\{0\}$ is nonsingular if and only if the $\boldsymbol{C}$-vector space

$$A(V)=\mathcal{O}_{n+1}/((f)+\Delta(f))$$

is finite dimensional, where $\Delta(f)$ is the ideal in $\mathcal{O}_{n+1}$ generated by the first partial derivatives of f. $A(V)$, provided with the obvious $\boldsymbol{C}$-algebra structure, is called the moduli algebra of V. In [4] the following theorem was proved.

Theorem 1 (Mather-Yau). *Suppose $(V, 0)$ and $(W, 0)$ are germs of hypersurfaces in $\boldsymbol{C}^{n+1}$, and $V-\{0\}$ is nonsingular. Then $(V, 0)$ is biholomorphically equivalent to $(W, 0)$ if and only if $A(V)$ is isomorphic to $A(W)$ as a $\boldsymbol{C}$-algebra.*

It is natural to raise the recognition problem: When a commutative local Artinian algebra is a moduli algebra? How can one construct the singularity $(V, 0)$ explicitly from the moduli algebra $A(V)$. In this short note, we shall answer the above questions in the case $(V, 0)$ is a weighted homogeneous singularity. We thank Herwig Hauser for encouraging us in writing up this note for publication.

Let A be a commutative Noetherian algebra with maximal ideal m. Let $x_1, \cdots, x_k$ be a system of minimal generating set of m such that their images in m/m^2 form a basis. Consider the algebra homomorphism

$$\varphi: \boldsymbol{C}\{z_1, \cdots, z_k\}\longrightarrow A$$

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