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# Regular System of Weights and Associated Singularities

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## Introduction

Let $(a, b, c; h)$ be a system of four positive integers, called the weights, such that $h > \max(a, b, c)$. We call $(a, b, c; h)$ regular, if the following rational function in $T$ becomes a polynomial function. ((1.2) Definition)

$$\frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}.$$

(Notably, then the function becomes a polynomial with *positive integral* coefficients. ((1.3) Theorems 1 and 1*))

The purpose of the present paper is the study of such *regular systems of weights* in certain good cases, since it gives a systematic viewpoint for certain class of discontinuous groups and associated surface singularities, which are studied by several authors such as Arnold, Brieskorn, Dolgachev, Looijenga, Milnor, Orlik, Pinkham, Saito, Sherbak, Slodowy, Wagreich aud Wahl in connection with $\boldsymbol{C}^*$-action.

Namely we put $\varepsilon := a + b + c - h$ and classify regular systems of weights for $\varepsilon = 1$ ($>0$), $\varepsilon = 0$ and $\varepsilon = -1$ ((2.1) Theorem 2 and Tables 1, 2 and 3). Then each regular system for these three cases corresponds to a certain discrete subgroup of the groups of the motions of $\boldsymbol{P}^1(\boldsymbol{C})$, $\boldsymbol{C}$, and $\boldsymbol{H}$ (the upper half plane) respectively (cf. (3.4) Note). The correspondence is established through surface singularities. Namely let $(a, b, c; h)$ be a regular system of weights. Then the hypersurface $X_0 := \{(x, y, z) \in \boldsymbol{C}^3 : f(x, y, z) = 0\}$ defined by a weighted homogeneous polynomial $f(x, y, z) = \sum a_{ijk} x^i y^j z^k$ (here the sum is over indexes $(i, j, k)$ with $ai + bj + ck = h$ with generic coefficients), has singularity only at the origin. ((3.2) Theorem 3). For the cases $\varepsilon = 1, 0$ or $-1$, the 2-manifold $X_0 - \{0\}$ is a quotient variety by the free action of the corresponding discrete group on the canonical, trivial or anti-canonical $\boldsymbol{C}^*$-bundle over $\boldsymbol{P}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ respectively. (Some of such description is very old, going back to Schwarz's

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