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On the Resolution of the Three Dimensional Brieskorn Singularities

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§ 1. Introduction

Let $f(z_0, \cdots, z_n)$ be a germ of an analytic function at the origin with an isolated critical point at $z=0$ and $f(0)=0$. We assume that the Newton boundary $\Gamma(f)$ is nondegenerate. Let $V=f^{-1}(0)$ and let Σ^* be a simplicial subdivision of the dual Newton diagram. Then there is a resolution $\pi: \tilde{V} \to V$ which is associated with Σ^*. For each strictly positive vertex P of Σ^* such that $\dim \Delta(P) \geq 1$, there is a corresponding exceptional divisor $E(P)$. The purpose of this paper is to study the above resolution and to study the geometry of $E(P)$ in the case that $n=3$ and $f(z)=z_0^{a_0}+z_1^{a_1}+z_2^{a_2}+z_3^{a_3}$ with P being the weight vector of f. In Section 2, we will recall basic notations and the construction of the resolution of $V=f^{-1}(0)$. In Section 3, we will prove an isomorphism theorem about the exceptional surface $E(P)$ (Theorem (3.6)) which is one of the main results of this paper. In Section 4, we give a necessary and sufficient condition about $a=(a_0, a_1, a_2, a_3)$ for $E(P)$ to be a rational surface or a $K3$-surface. (Theorem (4.1) and Theorem (4.2)). There are 14 cases for $E(P)$ to be a rational surface and 22 cases for $E(P)$ to be a $K3$-surface up to Theorem (3.6). In Section 5, we will give the proof of Theorem (4.1) and Theorem (4.2).

§ 2. Preliminaries

Let $f(z)=\sum_\nu a_\nu z^\nu$ be the Taylor expansion of f. The Newton polygon $\Gamma_+(f)$ is the convex hull of $\cup\{\nu+(R^+)^{n+1};\ a_\nu \neq 0\}$ and the union of its compact faces is denoted by $\Gamma(f)$ which is called the Newton boundary of f. Let N^+ be the set of the positive vectors of R^{n+1} which are considered to be in the dual space of R^{n+1} through the Euclidean inner product. For each $P \in N^+$, let $d(P)$ be the minimal value of $\{P(x);\ x \in \Gamma_+(f)\}$ and let $\Delta(P)=\{x \in \Gamma_+(f);\ P(x)=d(P)\}$. Two vectors P and Q in N^+ are said to be equivalent if and only if $\Delta(P)=\Delta(Q)$. The

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