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Configurations Related to Maximal Rational Elliptic Surfaces

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Introduction

The purpose of this paper is to present a method to attack the problem of finding good configurations of plane curves. Our viewpoint is described as follows: Take two cubic curves intersecting properly on the complex projective plane $P_2(C)$ and then form the pencil generated by them. When regarded as a one-parameter family of elliptic curves, the pencil defines an elliptic surface in the sense of [Kd] which is obviously blown down to $P_2(C)$. The singular fibers then correspond to exactly the singular member of the pencil. A global section is mapped by the blowing down to either a base point of the pencil or a curve which, outside the base points, intersects the generic member of the pencil at exactly one point. In many cases the singular members and the images of global sections might already form an interesting configuration of plane curves. But, by fixing a global section to give the group structure to fibers, we can further observe the locus of m-torsion points of fibers for every $m>0$ ($m \in Z$), which is also regarded as a plane curve. Adding some of these loci might in general enrich the original configuration.

The motivation to the present work comes from the following fact: Hirzebruch [H2] uses some arrangements of lines on $P_2(C)$ for the purpose of constructing compact quotients of the unit ball in C^2 in algebro-geometric way. A little earlier Inoue [In] and Livné [L] used the elliptic modular surfaces associated with some congruence subgroups of $SL(2, Z)$ for the same purpose (see [Sh] for this notion). But, some of these examples can actually be treated in a unified way from the viewpoint described above.

Now note that such quotients can not be deformed continuously while the moduli space of rational elliptic surfaces has eight parameters. Thus we have to impose the maximality assumption on our elliptic surface to make it rigid; this roughly amounts to assuming that the surface has only finitely many global sections.

The geometry of rational elliptic surfaces is equivalent to that of del Pezzo surfaces of degree one, the relation of which to the root system of

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