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Complex Analytic Singularities
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On the Equations $x^p+y^q+z^r-xyz=0$

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To the memory of the late Professor Takehiko Miyata

Introduction

We know two strange dualities—the duality of fourteen exceptional unimodular singularities [A] and the duality of fourteen hyperbolic unimodular singularities [N1]. The first purpose of this article is to recall and compare them. The second is to give explanations for the second duality from various viewpoints. The third is to study deformations of $T_{p,q,r}$, the singularity defined by the equation in the title, or more generally cusp singularities by means of hyperbolic Inoue surfaces (Inoue-Hirzebruch surfaces).

This article is organized as follows. In Section 1 we recall a basic notion of modality of singularities and a classification of zero and one modal hypersurface singularities. In Section 2 we recall the duality of exceptional unimodular singularities. In Section 3 we recall the duality of hyperbolic unimodular singularities. A remarkable fact is that both of the dualities take place for the same pairs of triples—the fourteen Dolgachev (or Gabrielov) triples. A typical pair of the second duality is

$$T_{3,4,4}:\quad x^3+y^4+z^4-xyz=0,$$
$$T_{2,5,6}:\quad x^2+y^5+z^6-xyz=0.$$

Sections 4–7 are devoted to studying the second duality. In Section 4 we recall hyperbolic Inoue surfaces and the duality of cycles of rational curves on them. The exceptional sets of $T_{3,4,4}$ and $T_{2,5,6}$ are cycles of rational curves and both cycles appear on one and the same hyperbolic Inoue surface. In Section 5 we shall give a number-theoretic explanation for the duality. We will see that the duality is essentially the relationship between a complete module and its dual in a real quadratic field. In Section 6 we shall provide a geometric explanation for the duality by means of general theory of surfaces of class VII_0. In Section 7 we shall

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