Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 101–124

On the Homotopy Theory of Arrangements

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In this paper an arrangement $\mathscr{A}$ is a finite collection of hyperplanes $\{H_1, \cdots, H_n\}$ through the origin in $\boldsymbol{C}^l$. We wish to examine the complementary space $M=\boldsymbol{C}^l-\bigcup_{i=1}^n H_i$ from a topological point of view. More specifically, we will discuss the homotopy properties of M, and how these properties relate to various other well-known properties of arrangements. As a focal point we will consider the question:

Precisely when is M a $K(\pi, 1)$ space?

Arrangements arise in many contexts. For example, one may refer to papers by Orlik, Sommese, and Terao in this volume. The question of when M is a $K(\pi, 1)$ was first considered by Fadell and Neuwirth [6], who gave an affirmative answer for the arrangements of type A_k (see (2.1) for definitions). Such questions burst upon the singularities scene with the work of Arnol'd and Brieskorn reported on in [3], and the lovely result of Deligne [5] that real simplicial (hence real reflection) arrangements (see (2.4)) yield $K(\pi, 1)$ spaces. In the time since that work a number of other properties of arrangements have been defined, some with the $K(\pi, 1)$ property in mind, and some in other contexts. We intend here to mention those properties which seem relevant and to try to sort out their interrelationships.

Since many of our readers will be familiar with most of these properties, we will defer precise definitions and examples until Section 2. We start in Section 1 with a broad overview of the field. Then after giving the relevant definitions we will consider each possible implication in a systematic fashion in Section 3. The section may be treated as a reference section, though it begins with a discussion of some major positive results and their proofs. Where counterexamples are required, we have tried to manage with as few as possible. All this information is assembled in a chart at the end of Section 2. A quick glance at this chart shows quite a number of question marks. In the final section we construct a commuta-

Received January 1, 1985.
Revised February 14, 1986.

* This paper is an expanded version of part of the first author's thesis [7], and is partly based on the talk given by the second author at the U.S.-Japan Seminar.

** Supported by NSF.