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Topological Restrictions on the Links of Isolated Complex Singularities

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This note is a preliminary report on joint work with Richard Hain; the full details will appear shortly [4]. We will sketch a proof of the following result:

Theorem 1. *Let L be the link of an isolated singular point of a complex algebraic variety X of dimension n. If $s, t<n$ and $s+t\geq n$, then* the *cup product*

$$H^s(L, Q)\otimes H^t(L, Q)\longrightarrow H^{s+t}(L, Q)$$

vanishes.

Recall that the *link* L of a singular point is by definition the intersection of the variety with a small sphere about that point. If the variety is n-dimensional and the singularity is isolated, then L is a real $(2n-1)$-manifold.

This theorem shows that there are restrictions on the topology of isolated singular points of complex varieties. For example, any manifold of the form $K\times M\times N$ where $\dim K<n$, $\dim M<n$, $\dim K+\dim M\geq n$ and $\dim K+\dim M+\dim N=2n-1$ cannot occur as such a link. The case $n=2$ of this theorem was shown in [14], where the Grauert contraction theorem was used. Apparently this is the first known such restriction in higher dimensions. In fact, this result gives a necessary condition for contraction. For example, $S^2\times S^2\times S^3$ cannot occur as the link of an isolated singular point, but it occurs as the link of $P^1\times P^1$ in $P^1\times P^1\times C^2$. Hence this set cannot be contracted to a point. The techniques used in this paper are similar to those used by Morgan to find an example of a homotopy type which cannot occur as a smooth algebraic variety. On the other hand, for any finite simplicial complex there is a quasi-projective variety (non-compact, with singularities) of the same homotopy type [1, Section 9].

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