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Introduction to the L^2-Cohomology of Arithmetic Quotients of Bounded Symmetric Domains

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Let

G = the group of real points on a semi-simple group $\boldsymbol{G}$ defined over $\boldsymbol{Q}$
Γ = a neat arithmetic subgroup of G (see beginning of § 3)
K = a maximal compact subgroup of G
$\mathfrak{X}$ = the symmetric space G/K, assigned a G-invariant Riemannian metric and assumed to possess a G-invariant complex structure.

It has been conjectured [29] that the L^2-cohomology of the quotient $V=\Gamma\backslash\mathfrak{X}$ is naturally isomorphic to the middle intersection cohomology [22] of the compactification V^* of V constructed by Satake [26] in certain cases and by Baily and Borel [2] in general. I recall that V^* is a projective, normal, but in general highly singular algebraic variety. This conjecture was verified by Zucker himself [29] for a few cases where $\boldsymbol{G}$ has rational rank one, by Borel for all remaining groups of rational rank one (an announcement appears in [4]), and by Zucker [30] for a few cases of rational rank two. Very recently Borel and I working together have concluded the proof for all rational rank two groups (an announcement [6] will appear soon).

On the one hand, the proof that Borel and I have concocted seems even to us extraordinarily complicated. On the other, large parts of it carry through for groups of arbitrary rational rank and it certainly looks plausible (to me, at least) that an extension of our techniques will eventually work in general. What I propose to do in this paper, therefore, is to explain bits and pieces of what we have done in a relatively informal way. Actually the paper may be divided into two relatively independent parts: the first two sections form a general introduction to L^2-cohomology, and the last three deal more particularly with Zucker's conjecture. There is much overlap in the first two sections with [15], but I have made more precise the connection between the obvious duality of L^2-cohomology and a corresponding, more technical, duality of associated complexes of sheaves.

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