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On the Symmetric-Square Zeta Functions Attached to Hilbert Modular Forms

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In this note we present new proofs of properties of the "second" L-functions attached to modular forms without using the Rankin-Selberg method. Detailed proofs (for Hilbert modular cases) are contained in [7].

§ 1. Elliptic Modular Case

Let

$$f(z)=\sum_{n=1}^{\infty} a(n)e(nz)$$

be a normalized eigen cusp form of weight k with respect to $SL(2, \boldsymbol{Z})$. Here k is a positive integer, $e(x)=\exp(2\pi ix)$, and z is a variable on the upper half plane $\mathfrak{H}$. The "second" L-function we consider here is defined by:

$$L_2(s,f)=\prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1}$$

where p runs over all prime numbers, and α_p, $\beta_p \in \boldsymbol{C}$ are taken so that $\alpha_p+\beta_p=a(p)$, $\alpha_p\beta_p=p^{k-1}$; this infinite product converges absolutely and uniformly for $\mathrm{Re}(s)>k$.

The following properties are known:

(i) (Shimura [9], Zagier [11], Gelbart-Jacquet [5])

$L_2(s,f)$ has a holomorphic continuation to the whole s-plane and satisfies a functional equation under $s\mapsto 2k-1-s$.

(ii) (Zagier [11], Sturm [10]) For each even integer m with $k\leq m \leq 2k-2$, the value $L_2(m,f)/\pi^{2m-k+1}(f,f)$ belongs to the totally real number field $\boldsymbol{Q}(f)=\boldsymbol{Q}(a(n)\,|\,n\geq 1)$; here (,) is the Petersson inner product (cf. (2.2) below).

Most of the known proofs of (i) (ii) depend on the Rankin-Selberg method. The main purpose of this note is to give proofs of (i) (ii) not using the Rankin-Selberg method. Poincaré series and Kloosterman sums

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