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On the Discriminant of Transformation Equations of Modular Forms

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Introduction

In our previous paper [3], we have proved that the transformation equations of certain modular forms can be expressed by special values of the zeta functions of those forms. At the symposium, we talked about this result. Here we give some results obtained after that.

Let f be a modular form of weight k on the congruence subgroup $\Gamma_0(p)$ of $SL_2(\boldsymbol{Z})$. We assume that *p is an odd prime throughout the paper.* Then the transformation equation of f is defined by

$$\Phi(X;f)=\prod_{\alpha\in\Gamma_0(p)\backslash SL_2(\boldsymbol{Z})}(X-f|_k\alpha)=0,$$

where $(f|_k\gamma)(z)=\det(\gamma)^{k/2}f((az+b)/(cz+d))(cz+d)^{-k}$ for

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2^+(\boldsymbol{R})=\{\gamma\in GL_2(\boldsymbol{R})\,|\det(\gamma)>0\}.$$

Obviously all coefficients σ_μ of $\Phi(X;f)$ are modular forms on $SL_2(\boldsymbol{Z})$, and therefore, the discriminant D of $\Phi(X;f)$ is also a modular form on $SL_2(\boldsymbol{Z})$ of weight $p(p+1)k$. We call that the transformation equation $\Phi(X;f)=0$ is *$\boldsymbol{Z}$-rational* if all coefficients σ_μ have $\boldsymbol{Z}$-rational Fourier expansions as modular forms (see § 1, for the $\boldsymbol{Z}$-rationality of Fourier expansions). Then one of our results is

Theorem 1. *If the transformation equation $\Phi(X;f)=0$ of f is $\boldsymbol{Z}$-rational and if p is an odd prime, then the discriminant D of $\Phi(X;f)$ is expressed as*

$$D=\begin{cases}(-1)^{(p-1)/2}p^p\Delta^{p+1}h^2, & \textit{if } f \textit{ is a cusp form,}\\ (-1)^{(p-1)/2}p^p\Delta^{p-1}h^2, & \textit{otherwise,}\end{cases}$$

where h is a modular form on $SL_2(\boldsymbol{Z})$ with a $\boldsymbol{Z}$-rational Fourier expansion and Δ is Ramanujan's function $\exp(2\pi iz)\prod_{n=1}^{\infty}(1-\exp(2\pi inz))^{24}$.

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