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On the Stark-Shintani Conjecture and Certain Relative Class Numbers

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§ 1. Introduction

1.1. In [4], [5], H. M. Stark introduced certain ray class invariants of real quadratic fields with the use of special values at $s=0$ of the derivatives of some zeta functions, and presented a remarkable conjecture on the arithmetic of the ray class invariants (his treatment covers the cases of totally real fields). T. Shintani established the conjecture independently and solved it in a special but non-trivial significant case (see [3]). The solved case of the conjecture owing to Shintani might be of some importance in connection with certain $\boldsymbol{Z}_p$-extensions of ray class fields over real quadratic fields (see J. Nakagawa [1], [2]). In this note we obtain a certain relative class number formula of the ray class fields under the assumption that the Stark-Shintani conjecture is valid. Such a class number formula will have some application in the study of $\boldsymbol{Z}_p$-extensions of the ray class fields (cf. [1], [2]).

1.2. We summarize our results. Let F be a real quadratic field embedded in the real number field $\boldsymbol{R}$. Let $E(F)$ (resp. $E^+(F)$) be the group of units (resp. totally positive units) of F. For each $\alpha \in F$, α' denotes the conjugate of α in F. For an integral ideal $\mathfrak{f}$ of F, let $H_F(\mathfrak{f})$ denote the group of narrow ray classes modulo $\mathfrak{f}$ of F. Take a totally positive integer ν of F such that $\nu+1 \in \mathfrak{f}$, and denote by $\nu(\mathfrak{f})$ the ray class of $H_F(\mathfrak{f})$ represented by the principal ideal (ν). For each class $c \in H_F(\mathfrak{f})$, let $\zeta_F(s, c)$ be the partial zeta function defined by $\zeta_F(s, c)=\sum N(\mathfrak{a})^{-s}$, where $\mathfrak{a}$ is taken over all integral ideals of F belonging to the class c. It is known that $\zeta_F(s, c)$ is holomorphic in the whole complex plane except for a simple pole at $s=1$. Let $\zeta'_F(s, c)$ denote the derivative of $\zeta_F(s, c)$. Set, for each $c \in H_F(\mathfrak{f})$,

$$X_{\mathfrak{f}}(c)=\exp\left(\zeta'_F(0, c)-\zeta'_F(0, c\nu(\mathfrak{f}))\right).$$

The invariant $X_{\mathfrak{f}}(c)$ is intensively studied by Stark [4] and Shintani [3].

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