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A Description of Discrete Series for Semisimple Symmetric Spaces

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§ 1. Introduction

Let G be a connected real semisimple Lie group, σ an involution of G, and H the connected component of the fixed-point group G^σ containing the identity. Then G/H is called a semisimple symmetric space ([1], [5]). We assume in this paper that G is a real form of a complex Lie group G_c. When G/H satisfies the condition

$$\operatorname{rank}(G/H)=\operatorname{rank}(K/K\cap H), \tag{1.1}$$

Flensted-Jensen [5] constructed countably many discrete series for G/H. Here K is a σ-stable maximal compact subgroup of G and "discrete series for G/H" are equivalence classes of the representations of G on minimal closed G-invariant subspaces in $L^2(G/H)$. In this paper we give a theorem that describes all the discrete series for G/H. Especially there is no discrete series when $\operatorname{rank}(G/H)\neq\operatorname{rank}(K/K\cap H)$.

The result of this paper can be described as follows.

Let $\mathfrak{g}$ be a semisimple Lie algebra and σ an involution ($\sigma^2=$identity) of $\mathfrak{g}$. Fix a Cartan involution θ such that $\sigma\theta=\theta\sigma$. Let $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ (resp. $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ). Let $\mathfrak{g}_c$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$ be subalgebras in $\mathfrak{g}_c$ defined by

$$\mathfrak{g}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q})+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h})+\mathfrak{p}\cap\mathfrak{q},$$
$$\mathfrak{k}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{p}\cap\mathfrak{h}),\qquad \mathfrak{h}^d=\mathfrak{k}\cap\mathfrak{h}+\sqrt{-1}(\mathfrak{k}\cap\mathfrak{q}).$$

Extend σ and θ to complex linear involutions of $\mathfrak{g}_c$. The restrictions of σ and θ to $\mathfrak{g}^d$ are denoted by the same letters. Then $(\mathfrak{g}^d, \mathfrak{k}^d, \mathfrak{h}^d, \sigma, \theta)$ satisfies the same condition as $(\mathfrak{g}, \mathfrak{k}, \mathfrak{h}, \theta, \sigma)$.

Let G_c be a connected complex Lie group with Lie algebra $\mathfrak{g}_c$, and let G, K, H, G^d, K^d, H^d, H_c and K_c be the analytic subgroups of G_c corresponding to $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$, $\mathfrak{g}^d$, $\mathfrak{k}^d$, $\mathfrak{h}^d$, $\mathfrak{h}_c$ and $\mathfrak{k}_c$, respectively. Let $\hat{K}$ (resp. $\hat{H}^d$)

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