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Initial Value Problem for the Toda Lattice Hierarchy

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§ 1. Introduction

A few years ago there appeared several interesting attempts [1–4] to apply some group theoretical point of view to the explicit integration of the Toda lattice. In this direction, however, no result seems to have been established for the infinite lattice without free ends or any periodicity.

This paper presents an algebraic approach toward the integration of the infinite Toda lattice which, in general, does not fall into the cases discussed in [1–4]. As a result, the initial value problem for the Toda lattice hierarchy [5] is explicitly solved.

To set up the initial value problem, let us briefly review the Toda lattice hierarchy [5]:

Let $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ be independent variables with infinite many components, and L, M matrices of size $\boldsymbol{Z}\times\boldsymbol{Z}$ ($\boldsymbol{Z}$ denotes the totality of integers) of the form

$$(1.1)\qquad \begin{aligned} &L=(b_{j-i}(i, x, y))_{ij\in \boldsymbol{Z}}, \quad b_j=0\ (j>1), \quad b_1=1, \\ &M=(c_{j-i}(i, x, y))_{ij\in \boldsymbol{Z}}, \quad c_j=0\ (j<-1), \quad c_{-1}\neq 0. \end{aligned}$$

b_j and c_j serve as the unknown functions of the nonlinear differential equations describing the Toda lattice hierarchy. Auxiliary matrices B_μ, C_μ, $\mu=1, 2, \cdots$, are introduced by

$$(1.2)\qquad B_\mu=(L^\mu)_+, \qquad C_\mu=(M^\mu)_-,$$

where the symbols $(A)_\pm$ denote for a matrix $A=(a_{ij})_{ij\in\boldsymbol{Z}}$ of size $\boldsymbol{Z}\times\boldsymbol{Z}$ the triangular matrices $(a_{ij}Y^\pm_{j-i})_{ij\in\boldsymbol{Z}}$, respectively, with $Y_s^+=0$ $(s<0)$, $=1$ $(s\geq 0)$, $Y_s^-=1$ $(s<0)$, $=0$ $(s\geq 0)$.

The Toda lattice hierarchy is defined by the system of the Lax type

$$(1.3)\qquad \begin{aligned} &\partial_{x_\mu}L=[B_\mu, L], \quad \partial_{y_\mu}L=[C_\mu, M], \\ &\partial_{x_\mu}M=[B_\mu, M], \quad \partial_{y_\mu}M=[C_\mu, M], \quad \mu=1, 2, \cdots, \end{aligned}$$

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