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Maximal Functions on Non-Compact, Rank One Symmetric Spaces

Radial Maximal Functions and Atoms

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§ 1. Introduction

As in the case of a Euclidean space, the theories of maximal functions, Hardy spaces, atoms, etc. are rapidly expanding in the case of homogeneous groups, that is, connected and simply connected nilpotent Lie groups whose Lie algebras are endowed with a family of dilations (see G. B. Folland and E. M. Stein [5]) and more generally, spaces of homogeneous type (see R. R. Coifman and G. Weiss [3, § 2]). In this paper we shall attempt to develop these theories on a rank one, irreducible Riemannian symmetric space of non-compact type (see S. Helgason [8, Chs. V and VI]). Of course such a space is not of homogeneous type.

Let G be a connected, real rank one semisimple Lie group with finite center and $G=KAN$ an Iwasawa decomposition for G. Put $X=G/K$. Then X has a G-invariant measure $d\mu$ (resp. a metric) induced by the G-invariant measure dg on G (resp. the Killing form of the Lie algebra of G). For each locally integrable function f on X, the Hardy-Littlewood maximal function $M_{HL}f$ is defined by

$$M_{HL}f(x)=\sup_{0<\varepsilon<\infty}|B(x,\varepsilon)|^{-1}\int_{B(x,\varepsilon)}|f(g)|d\mu(g)\quad(x\in X), \tag{1.1}$$

where $B(x, \varepsilon)$ is the open ball on X around x with radius ε and $|B|$ is the volume of the ball. Then J. L. Clerc and E. M. Stein [1] and Jan-Olov Strömberg [12] showed that the operator M_{HL} is of type (L^p, L^p) for $p>1$ and of weak type (L^1, L^1) respectively. Now we shall define a radial maximal function as an extension of $M_{HL}f$ as follows. Since A is one dimensional, we can parametrize elements of A as a_t $(t\in \boldsymbol{R})$ and express the Cartan decomposition of x in G as $x=k_1a_{t(x)}k_2$ $(k_1, k_2\in K$ and $t(x)\geqq 0)$. Put $\Delta(t)=(\operatorname{sh} t)^{m_1}(\operatorname{sh} 2t)^{m_2}$, where m_1 and m_2 are the multiplicities of the roots α and 2α of (G, A) respectively. Then for f in $C_c^\infty(G)$,

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