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Irreducible Decomposition of Fundamental Modules for $A_l^{(1)}$ and $C_l^{(1)}$, and Hecke Modular Forms

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§1.

In [1], [2] Kac and Peterson established a connection between the characters of irreducible highest weight modules over Euclidean Lie algebras and the classical theta and modular functions. In particular, they considered the generating functions of weight multiplicities along the direction of multiples of the null root (the string functions), and derived their automorphic properties.

In this paper, we treat a different sort of generating function which we encounter when we consider the decomposition of a highest weight module with regards to a subalgebra. To be specific, let us consider $A_{2l-1}^{(1)}$ and its subalgebra $C_l^{(1)}$. Let $\tilde{\Lambda}_j$ $(0 \leqq j \leqq 2l-1)$ denote the fundamental weights for $A_{2l-1}^{(1)}$ and let $L(\tilde{\Lambda}_j)$ denote the associated highest weight module. We use Λ_k and $L(\Lambda_k)$ $(0 \leqq k \leqq l)$ to denote those with respect to $C_l^{(1)}$. When we consider $L(\tilde{\Lambda}_j)$ as a $C_l^{(1)}$ module, it is no longer irreducible, but is decomposed into irreducible parts, each of which is isomorphic to one of $L(\Lambda_k)$'s. Thus, in terms of the characters, we have the identity of the form

$$\mathrm{ch}_{L(\tilde{\Lambda}_j)}|_{\mathfrak{h}} = \sum_{k=0}^{l} E_{jk}^{l}(q)\, \mathrm{ch}_{L(\Lambda_k)}.$$

Here $|_{\mathfrak{h}}$ means the restriction to the Cartan subalgebra of $C_l^{(1)}$, and $q = e^{-\delta}$ where δ is the null root of $C_l^{(1)}$. Our problem is to determine the power series $E_{jk}^{l}(q)$. If we set

$$e_{jk}^{l}(\tau) = q^{(j-k)/2 - j^2/4l + (k+1)^2/4(l+2)} E_{jk}^{l}(q)$$

with $q = e^{2\pi i\tau}$, it follows from the result of Kac and Peterson that $e_{jk}^{l}(\tau)$ is a modular form. On the other hand, through computer experiments we found the following asymptotic property as $l \to \infty$: there exist power series $F_{jk}^{(\nu)}(q)$ $(\nu = 0, 1, 2, \cdots)$ *independent of* l such that

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