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A Geometric Significance of Total Curvature on Complete Open Surfaces

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1. Let M be a 2-dimensional complete non-compact Riemannian manifold with non-negative Gaussian curvature K. Then the total curvature of M satisfies the inequality

$$\int_M K\,dv \leqq 2\pi,$$

where dv is the volume element of M induced from the Riemannian metric on M. This was proved by Cohn-Vossen in [2]. Obviously in contrast with compact case, the total curvature of M is not a topological invariant when M is non-compact and it depends on the Riemannian structures on M. Concerning this fact, in [5], [7], we showed that the total curvature of M is expressing a certain curvedness of M. We will state it in the following.

For a point $q \in M$, put $S_q(M) := \{v \in T_q(M);\ \text{norm of } v=1\}$, where $T_q(M)$ is the tangent space of M at q. From the Euclidean metric on $T_q(M)$, $S_q(M)$ becomes a Riemannian submanifold of $T_q(M)$ isometirc to the standard unit circle. Thus we can consider the Riemannian measure on $S_q(M)$. Let $A(q) \subset S_q(M)$ be the set defined as

$$\{v \in S_q(M);\ \text{geodesic } \gamma\colon [0,\infty) \to M \text{ given by } \gamma(t) = \exp_q tv \text{ is a ray}\}.$$

Here $\exp_q\colon T_q(M) \to M$ is the exponential mapping of M and geodesic γ is called a ray when any subarc of γ is a shortest connection between its end points. Using these notations, the facts mentioned above are stated as follows.

Fact 1. Let M be a 2-dimensional complete Riemannian manifold with non-negative Gaussian curvature K diffeomorphic to a Euclidean plane. Then for any point $q \in M$,

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