Geometry of Geodesics and Related Topics
Advanced Studies in Pure Mathematics 3, 1984
pp. 423–450

Topology of Complete Noncompact Manifolds

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§ 0. Introduction

A well known theorem due to Cohn-Vossen [8] states that if an oriented complete, noncompact and finitely connected Riemannian manifold M of dimension 2 admits the total curvature $c(M)=\int_M G\,dM$, where G is the Gaussian curvature and dM is the volume element of M, then $c(M)\leqq 2\pi\chi(M)$. From this fact he proved that if M has nonnegative Gaussian curvature, then M is either diffeomorphic to a plane R^2 or else isometric to a flat cylinder $S^1\times R$ or a flat open Möblius strip. This pioneering work of Cohn-Vossen has been extended by Cheeger, Gromoll and Meyer in [7], [20] and others to obtain the structure theorem for complete noncompact Riemannian manifolds of nonnegative sectional curvature. The structure theorem states that if a complete noncompact Riemannian manifold M has nonnegative sectional curvature, then there exists a compact totally geodesic submanifold S of M (which is called the soul of M and has dimension $\geqq 0$) such that M is homeomorphic (or even diffeomorphic, see [33]) to the total space of the normal bundle $\nu(S)$ over S in M. The proof is done by constructing a family of compact totally convex sets exhausting M. It turns out that this family of compact totally convex sets is nothing but the sublevels of a convex function which is obtained by Busemann functions for rays emanating from an arbitrary fixed point.

Thus Busemann functions play an essential role in the study of complete noncompact Riemannian manifolds. This function has been introduced by Busemann (see Section 22 in [2]) in order to establish a theory of parallels for straight lines on a straight G-space on which every two points are joined by a unique distance realizing geodesic.

One of the purposes of this survey note is to study fundamental theorems on complete noncompact Riemannian manifolds of nonnegative curvature which has been obtained by Cheeger, Gromoll Meyer and To-

Received January 8, 1983.
Revised April 1, 1983.
* Dedicated to Professor I. Mogi on his 60th birthday