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On the Green Function of a Complete Riemannian or Kähler Manifold with Asymptotically Negative Constant Curvature and Applications

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Introduction

In this paper we shall consider a complete noncompact Riemannian or kähler manifold whose curvature tensor is asymptotically close to that of the real or complex space form of negative curvature. Examples of such a manifold are supplied by strictly convex bounded domains with a certain metric in R^n and by strictly pseudoconvex bounded domains with the Bergman metric in C^n (see § 5 and Appendix B).

Our main concern is to find an asymptotic estimate of the Green function of such a manifold. The result is that it behaves just like the Green function of space forms (Theorems 2, 4).

As an application we give a differential geometric proof of Malliavin's estimate ([22]) of the Green function of a strictly pseudoconvex bounded domain relative to the Bergman metric (Corollary 1 in § 5). Namely, let D be such a domain with the smooth boundary ∂D and $G(p, q)$ be the Green function. Fix a point q. Then for some constants c_i, the inequalities

$$c_1 d_E(p, \partial D)^n \leqq G(p, q) \leqq c_2 d_E(p, \partial D)^n$$
$$|\nabla_p G(p, q)| \leqq c_3 d_E(p, \partial D)^n$$

are valid for all p away from q. Here $d_E(p, \partial D)$ is the euclidean distance to ∂D. Unfortunately our proof needs some assumption on the metric, which probably restricts the topological type of the domain.

Another application in the *real* case is to construct bounded harmonic functions (Corollary 2 in § 6). For that purpose we will give a geometric description of the Martin boundary and solve the Dirichlet problem for harmonic functions relative to this boundary (Theorem 8). In this case the curvature is assumed to be strictly negative and asymptotically negative

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