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# Applications of Laplacian and Hessian Comparison Theorems

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## § 0. Introduction

Rauch [41] proved a fundamental theorem on the lengths of Jacobi fields, called the Rauch comparison theorem. After him, Berger [4], Warner [48] and Heintze and Karcher [27] etc. had extended the Rauch comparison theorem. Especially, Heintze and Karcher showed a very general comparison theorem for the length and volume distortion of the normal exponential map of a submanifold. The proof of their comparison theorem in turn tells us some useful informations about the "*local*" behaviour of the Laplacian and Hessian of the distance function to a submanifold (cf. Greene and Wu [25] in the case when a submanifold is a point). On the other hand, Wu [49] has proved that, in certain situations, the Laplacian and the Hessian of a distance function in an appropriate weak sense can be "*globally*" estimated from above (cf. also Calabi [10], Cheeger and Gromoll [13, 14], Yau [51]). Moreover, making use of the method by Wu, we have shown in [32] general comparison theorems on the Laplacian and the Hessian of a distance function. The purpose of the present paper is to give several applications of our comparison theorems.

**0.1.** We shall first describe our Laplacian and Hessian comparison theorems. Let $M$ be a Riemannian manifold with (possibly empty) boundary $\partial M$. We write $M_0$ for the interior of $M$ ($M=M_0$ if $\partial M=\phi$). Let $X$ be a smooth vector field on $M$. We consider the second order elliptic operator $L_X=\Delta+X$ acting on functions, where $\Delta$ denotes the Laplace operator $\left(\text{i.e., locally } \Delta=\sum \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}\left(\sqrt{G}g^{ij}\frac{\partial}{\partial x^j}\right)\right)$. For a semi-continuous function $\varphi$ on a neighborhood of a point $x$ in $M$, an extended real number $S_X\varphi(x)$ is defined by

$$S_X\varphi(x):=\liminf_{r\to 0}\left\{\int_{\partial B_r(x)} -\frac{\partial G_r(x,\xi)}{\partial\nu(\xi)}\varphi(\xi)d\xi-\varphi(x)\right\}\Big/\int_{B_r(x)} G_r(x,y)dy,$$

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