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Convexity in Riemannian Manifolds without Focal Points

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§ 0. Introduction

Throughout this paper let M' be a complete Riemannian manifold and let a geodesic $\alpha: (-\infty, \infty) \to M'$ be parametrized by its arc-length. M' is said to have no *focal points* if every geodesic $\alpha: (-\infty, \infty) \to M'$ has no focal points as a 1-dimensional submanifold in M'. In this paper we shall deal with complete Riemannian manifolds N without focal points from the point of view of geometry of geodesics. In particular, we shall investigate relations between the existence of totally convex sets in such a manifold N (or convex functions on N) and the topological and metric structure of N.

However our starting point of the study is different from usual ones. In a paper of O'Sullivan [20] we find a nice exposition of having no focal points. Namely, he has stated that (1) M' has nonpositive sectional curvature if and only if $\langle Y, Y\rangle'' \geqq 0$ for every Jacobi field along every geodesic α, (2) M' has no focal points if and only if $\langle Y, Y\rangle' > 0$ for $t > 0$ where Y is any nontrivial Jacobi field along any geodesic vanishing at $t=0$, (3) M' has no conjugate points if and only if $\langle Y, Y\rangle > 0$ for $t > 0$ where Y is any non-trivial Jacobi field along any geodesic α vanishing at $t=0$. Therefore if M' has nonpositive sectional curvature, then it has no focal points, and if M' has no focal points, then it has no conjugate points. These three classes of Riemannian manifolds are actually distinct as was shown in Gulliver [16]. From this fact it is a natural question to ask whether there exists a condition which define a new class of Riemannian manifolds relative to the classification of these three classes. The condition in (1) means that $\|Y\|^2$ is a convex function for $t \in \boldsymbol{R}$. Near peaklessness of a function which is explained by Busemann-Phadke [5] is weaker than convexity. A continuous function f on $\boldsymbol{R}$ is by definition *nearly peakless* if $f(t_2) \leqq \mathrm{Max}\,\{f(t_1), f(t_3)\}$ for any $t_1 < t_2 < t_3$. A differentiable nearly peakless function f on $\boldsymbol{R}$ has a property: $f''(t) \geqq 0$ for those t such that $f'(t) = 0$.

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