Advanced Studies in Pure Mathematics 3, 1984
Geometry of Geodesics and Related Topics
pp. 283–309

Some Morse Theoretic Aspects of Holomorphic Vector Bundles

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§ Introduction

In this paper we shall consider certain theory of stationary points or loci arising from families of holomorphic sections of holomorphic vector bundles. The notion of Morse functions will be generalized to families of holomorphic sections, called quasilinear sections or holomorphic sections in quasilinear position. One of our results shows that this kind of sections exist generically in certain cases. We shall define a particular subset of Schubert cycles, called stationary loci, associated to quasilinear holomorphic sections. We are also concerned with a relation between these loci and characteristic classes. As Morse functions give us some information of topology of differentiable manifolds, it will turn out that our loci tell us some complex analytic structure of complex manifolds.

Let M be a compact complex manifold and let $E \to M$ be a holomorphic vector bundle of rank q. We denote by $\oplus^r \Gamma(M, \mathcal{O}(E))$ the set of all the families of holomorphic r-sections $\{\sigma_1, \cdots, \sigma_r\}$ of $E \to M$, $r \leq q$. Topology of the set $\oplus^r \Gamma(M, \mathcal{O}(E))$ is naturally defined taking into consideration higher order differentials. For the definition of quasilinear sections, see Section 1. Our generic existence theorem is stated as follows.

Generic existence Theorem. *Let M be a compact complex manifold and let $E \to M$ be a holomorphic vector bundle of rank q such that each fibre is generated by global holomorphic sections. Then, for any integer $r \leq q$, the set of all families of holomorphic sections $\{\sigma_1, \cdots, \sigma_r\}$ in quasilinear position forms an open and dense subset in $\oplus^r \Gamma(M, \mathcal{O}(E))$.*

Let $\sigma_1, \cdots, \sigma_r$ be holomorphic sections of $E \to M$. The Schubert cycle denoted by $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ is defined to be the subset of M consisting of points where $\sigma_1, \cdots, \sigma_r$ fail to be linearly independent (see § 2). If $\sigma_1, \cdots, \sigma_r$ are in quasilinear position, then it follows that the Schubert cycle $\mathscr{S}(\sigma_1, \cdots, \sigma_r)$ has only singularities of quasilinear type (see Def. 1.1).

Received December 28, 1982.