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On the Manifolds of Periodic Geodesics

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Let S^n be the n-dimensional sphere with a Riemannian metric g. If all geodesics are periodic with the same period l, we say that the Riemannian manifold (S^n, g) is a C_l-manifold, or g is a C_l-metric on S^n. (For detail see Besse [2]). Let g_0 be the canonical metric of S^n. Then (S^n, g_0) is a $C_{2\pi}$-manifold. There are some examples of C_l-metric on S^n (Zoll [10], Weinstein [2], Guillemin [5]) other than the canonical metric. These examples are all obtained from deformations of g_0 in the space of C_l-metrics.

Let $T_1(S^n, g) = T_1(S^n)$ denote the tangent sphere bundle of radius 1 of a C_l-manifold (S^n, g). Then the geodesic flow induces a free S^1-action on T_1S^n. Since the geodesic flow vector field is a contact vector field on T_1S^n, the quotient space T_1S^n/S^1 is a $(2n\text{-}2)$-dimensional symplectic manifold. We call T_1S^n/S^1 the manifold of geodesics and denote by $\mathrm{Geod}(S^n, g)$. The manifold $\mathrm{Geod}(S_n, g_0)$ is symplectically diffeomorphic to the Kähler manifold Q^{n-1}, called hyperquadric and defined by the equation

$$Z_0^2 + Z_1^2 + \cdots + Z_n^2 = 0$$

in CP^n. Since every known example of C_l-manifold (S^n, g) is a deformation of (S^n, g_0), the manifold of geodesics $\mathrm{Geod}(S^n, g)$ for such manifold is symplectically diffeomorphic to Q^{n-1}.

A result of Weinstein [7] says that, if $\mathrm{Geod}(S^n, g_1)$ and $\mathrm{Geod}(S^n, g_2)$ are symplectically diffeomorphic, then the eigenvalues of the Laplacian on two C_l-manifolds (S^n, g_1) and (S^n, g_2) are asymptotically similar.

Our problem is as follows. For any C_l-metric g on S^n, is $\mathrm{Geod}(S^n, g)$ diffeomorphic to Q^{n-1}? In this paper we study the tangent bundle of $\mathrm{Geod}(S^n, g)$ and its characteristic classes.

Since $Sp(n-1, \boldsymbol{R})$ is homotopy equivalent to $U(n-1)$, the symplectic manifold $\mathrm{Geod}(S^n, g)$ has the unique almost complex structure up to homotopy.

Let γ denote the complex line bundle associated to the S^1-principal

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