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A Differentiable Sphere Theorem for Volume-Pinched Manifolds

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Dedicated to Professor I. Mogi on his 60th birthday

§ 0. Introduction

A main problem in differential geometry is to investigate the influences of geometrical quantities of complete Riemannian manifolds on the topology. The sphere theorem due to Klingenberg states that if M is a complete simply connected manifold with the sectional curvature K_M, $1/4 < K_M \leqslant 1$, then M is a topological sphere ([7]). A stronger assumption for curvature implies that M is diffeomorphic to the standard sphere ([4], [8], [10]). In the proof of these results, an estimate of the injectivity radius $i(M)$, $i(M) \geqslant \pi$, of the exponential map on M plays an essential role. On the other hand, by pinching the diameter $\mathrm{diam}(M)$ in place of the sectional curvature Grove-Shiohama has obtained the following theorem which generalizes the Klingenberg sphere theorem.

Theorem A ([6]). *If the sectional curvature and the diameter of a complete manifold M satisfy $K_M \geqslant 1$, $\mathrm{diam}(M) > \pi/2$, then M is a topological sphere.*

Recently by pinching the volume $\mathrm{Vol}(M)$, Shiohama has proved the following sphere theorem for manifolds M of positive Ricci curvature Ric_M. We denote by S^n the unit n-sphere.

Theorem B ([9]). *For given n, $-\Lambda^2$, there exists an $\varepsilon = \varepsilon(n, \Lambda)$ such that if a complete manifold M of dimension n satisfies*

$$\mathrm{Ric}_M \geqslant 1,\ K_M \geqslant -\Lambda^2,\ \mathrm{Vol}(M) \geqslant \mathrm{Vol}(S^n) - \varepsilon,$$

then M is a topological sphere.

But in the situation of Theorem A or Theorem B, it was not known for M to be diffeomorphic to the standard sphere. The purpose of this

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