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Comparison and Finiteness Theorems in Riemannian Geometry

Takashi Sakai

This is a survey article on the above subject. A differentiable manifold admits variety of riemannian structures but we don't know in general what is the most adapted metric to the given differentiable structure. On the other hand, in riemannian geometry we have many important riemannian invariants, e.g., curvatures, volume, diameter, eigenvalues of Laplacians etc., and we know what is the most standard riemannian mani folds (model spaces) in terms of riemannian invariants, e.g., spaces of constant curvature, symmetric spaces, Einstein spaces etc.

We ask here the following problem: if riemannian manifolds are similar to the model spaces with respect to the riemannian invariants, are they also topologically similar?

This is in fact a kind of perturbation problem, but perturbation in terms of riemannian invariants and manifolds may vary during the perturbations. A typical example is the Hadamard-Cartan theorem which states that a complete simply connected riemannian manifold of non-positive curvature is diffeomorphic to the euclidean space. This follows from the fact that geodesic behavior from a point of the manifolds is similar to that of euclidean space. Namely the exponential map gives a diffeomorphism (see e.g. [B-C], [C-E], [G-K-M], [N-K], [K 6], [B 5]). Also many results from the theory of surfaces of fixed signed Gaussian curvature and the theory of space forms of constant curvature motivated such a question.

In 1951 H. E. Rauch proposed the above problem for sphere case and showed that if for sectional curvature K of a compact simply connected riemannian manifold $\min K/\max K$ is sufficiently close to 1, then the manifold is homeomorphic to the sphere. This was further developed by Berger, Klingenberg, Toponogov, Tsukamoto, Cheeger, Gromoll, Shiohama, Karcher, Ruh and other people and their works gave much influence on riemannian geometry. In Chapter 2 we treat the above problem.

On the other hand we may ask more generally: classify all the topological types of riemannian manifolds some of whose riemannian invariants satisfy some conditions. For instance classify manifolds of positive (or

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