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Complete Integrability of the Geodesic Flows on Symmetric Spaces

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Introduction

The geodesic flow of a Riemannian manifold M of dimension n is said to be *completely integrable* if there exist real-valued C^∞-functions $f_1, \cdots, f_n$ on the cotangent bundle T^*M which satisfy

(i) f_1 is the function assigning to each cotangent vector the square of its length,

(ii) $\{f_i, f_j\}=0$ for all $1 \leqq i, j \leqq n$, where $\{ , \}$ is the Poisson bracket, and

(iii) the set of critical points of the map $(f_1, \cdots, f_n)\colon T^*M \rightarrow \boldsymbol{R}^n$ has Liouville measure 0 in T^*M (cf. Def. 5.2.20 in [1]).

The classical examples of compact Riemannian manifolds with completely integrable geodesic flow are (a) compact surfaces of revolution, (b) $SO(3)$ with left invariant metric, (c) n-dimensional ellipsoids with different principal axes and (d) flat tori (cf. [5]). It is also known that the geodesic flow of a Zoll surface, which is not necessarily a surface of revolution, is completely integrable. In [6], Weinstein showed that the geodesic flow of the n-dimensional sphere S^n of constant curvature is completely integrable. Recently, Thimm showed that the geodesic flows of the following homogeneous spaces are completely integrable (cf. [4], [5]): (a) $G_{p,q}(\boldsymbol{R})$, (b) $G_{p,q}(\boldsymbol{C})$ (i.e., real and complex Grassmannians), (c) $SU(n)/SO(n)$, (d) a distance sphere in $P^n(\boldsymbol{C})$, (e) $SO(n)/SO(n-2)$. The method, exposed by Thimm, which allows the construction of families of first integrals in involution, is available for other homogeneous spaces.

In the present paper, we show that the geodesic flows of the following symmetric spaces are completely integrable: (a) $SU(n)$, (b) $SO(n)$, (c) $SU(2n)/Sp(n)$, (d) $SO(2n)/U(n)$. The procedure of the proof is as follows: Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ its dual space. The set $C^\infty(\mathfrak{g}^*)$ of all C^∞-functions on $\mathfrak{g}^*$ has a naturally defined Poisson structure (§ 1). In the case that $\mathfrak{g}$ is the Lie algebra of $SU(n)$ or of $SO(n)$, we construct concretely a commutative Poisson subalgebra $\Lambda(\mathfrak{g})$ of $C^\infty(\mathfrak{g}^*)$ and a system $\{F_j^{(i)}\}$ of its

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