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Integrability of Infinitesimal Zoll Deformations

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1. A Riemannian metric on a sphere S^n ($n \geqq 2$) is called a *Zoll metric* when all the geodesics are closed and have a common length 2π. The metric of constant sectional curvature 1 is a well-known example of a Zoll metric, but we further know that this standard metric g_0 is deformable by Zoll metrics (Zoll [8], Guillemin [3]; see also Besse [1]).

A symmetric 2-form h on S^n which is a direction of a Zoll deformation of g_0 satisfies

$$\int_0^{2\pi} h(\dot{\gamma}_0(s), \dot{\gamma}_0(s))ds = 0 \tag{1.1}$$

for every geodesic γ_0 of g_0 parametrized by its arclength s, where $\dot{\gamma}_0$ is the tangent vector of γ_0. Conversely, if h is a symmetric 2-form satisfying (1.1) for every geodesic of g_0, then the geodesics of $g_t = g_0 + t \cdot h$ are nearly 2π-periodic in the first order of t. We call such a symmetric 2-form on S^n an *infinitesimal Zoll deformation*, which we abbreviate as *IZD*. We say an *IZD* h is *integrable* if there exists a family of Zoll metrics g_t with g_0 being the standard one such that $h = \partial g_t/\partial t|_{t=0}$.

V. Guillemin proved in [3] that every *IZD* on a 2-dimensional sphere is integrable. On the other hand, K. Kiyohara ([4], [5]) showed that the situation is quite different in higher dimensions; not all the *IZD* are integrable, and, moreover, the set of integrable *IZD* does not even form a linear subspace.

They both studied the *IZD* of conformal type. Up to trivial *IZD*, they are the only possible *IZD* on S^2 (Funk [2]). But there exists another type of *IZD* in higher dimensions, as we have seen in [7]. In this paper, we shall exhibit that this type of *IZD* are not integrable, using a representation theoretical counterpart of Kiyohara's argument. The problem to determine which *IZD* is integrable is not yet resolved for the mixture of these two types of *IZD*, though we get some information by our argument.

2. We first recall how the condition (1.1) is deduced. Let g_t be a family of metrics on S^n with g_0 being the standard metric. We fix a point

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